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Banach Spaces of Distributions Having Two Module Structures

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After a discussion of a space of test functions and the corresponding space of distributions, a family of Banach spaces $(B, \| \|_B)$ in standard situation is described. These are spaces of distributions having a pointwise module structure and also a module structure with respect to convolution. The main results concern relations between the different spaces associated to B established by means of well-known methods from the theory of Banach modules, among them B_0 and $\tilde{B}$, the closure of the test functions in B and the weak relative completion of B, respectively. The latter is shown to be always a dual Banach space. The main diagram, given in Theorem 4.7, gives full information concerning inclusions between these spaces, showing also a complete symmetry. A great number of corresponding formulas is established. How they can be applied is indicated by selected examples, in particular by certain Segal algebras and the A_p-algebras of Herz. Various further applications are to be given elsewhere.

0. INTRODUCTION

While proving results for $L^p(G)$, $1 \leqslant p < \infty$ for a locally compact group G, one sometimes does not need much more information about these spaces than the fact that $L^p(G)$ is an essential module over $C^0(G)$ with respect to pointwise multiplication and an essential $L^1(G)$-module for convolution. It turns out that similar methods, suitably refined, can be applied to obtain related results for other Banach spaces B of "smooth" functions or "distributions," as long as they allow multiplications by a sufficiently rich family of smooth function and have a sufficiently large class of regularizing (convolution) operators acting on them. For example, as will be shown it is

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possible to approximate any $f \in B$ by test functions of the form $k * k' * hf$ in such spaces (under suitable hypotheses).

Although such a double module structure on B (one with respect to pointwise multiplication, the other being the convolutive structure) is frequently available for spaces considered in harmonic analysis (such as $L^p(G)$, $M(G)$, $A(G)$, $P(G)$, $A_p(G)$ in the sense of Herz, strongly character invariant Segal algebras (cf. [11]), solid, reflexive BF-spaces, etc.) or in the theory of functions spaces (Besov-spaces, Bessel-potentials, $F^s_{p,q}$-spaces, cf. [27], etc.), usually only implicit use is made of it. Thus the relevance of the presence of a pointwise structure in the discussion of spaces of multiplier (i.e., operators commuting with translations), and vice versa the implications due to the presence of a convolutive structure in the discussions of spaces of pointwise multipliers is often somewhat hidden. It is one of the purposes of this paper to point out the relevance of such structures, and in particular, to discuss their interdependence. As will be seen the connections and the symmetry between the two structures is quite strong (cf., for example, the main diagram given in Theorem 4.7).

Since we did not want to exploit this symmetry only for simple special cases we have decided to describe a general frame (to be called "standard situation") first, which of course requires the treatment of various technical problems. That the family of Banach spaces $(B, \| \|_B)$ in standard situation is large is indicated by the list of examples given in [12, Sect. 1]. Besides this fact the symmetry of this general frame allows to obtain results pairwise (cf. Section 4), i.e., an assertion and its "dual" version. Nevertheless it is sufficient to discuss one version in detail, the other version following by "dualisation," i.e., by interchanging the roles of multiplication and convolution. For certain Banach spaces of quasi-measures the dual version can be obtained by an application of the extended Fourier transform [11]. Furthermore, the sum or the intersection, as well as any interpolation space of a pair (B^1, B^2) of such spaces is again in standard situation, as well as the Banach dual B' of a minimal space B. Thus, applying the usual functional-analytic constructions (with some care) one does not leave the class of spaces under consideration.

This paper is organized as follows: In Section 1 the relevant definitions and basic results concerning Banach modules and harmonic analysis are given. In Section 3 the so-called standard situation is described: The spaces B to be considered in the sequel will be a Banach spaces of distributions having two module structures, one with respect to pointwise multiplication and the other with respect to convolution. In order to obtain full generality a description of the possible spaces A_0 of test functions and their topological dual is required. This is given in Section 2. It will allow to apply the results not only to spaces of (tempered) distributions on $\mathbb{R}^n$, but also to spaces (of kernels) of multipliers over locally compact groups or spaces of ultradistributions on $\mathbb{R}^n$ (cf. [3, 27]). It should be mentioned here that, nevertheless, the (almost) complete symmetry between these two structures can already be verified if one considers suitable Banach spaces of locally integrable functions, containing $\mathscr{K}(G)$ as a dense subspace, having isometric left translations and being Banach modules over $C^0(G)$ (pointwise), e.g. $C^0(G)$ itself. Having this situation in mind (corresponding to the choice $A = C^0(G)$, $A_0 = \mathscr{K}(G)$, $A_0' = R(G)$, i.e., space of Radon measures) would allow the reader to appreciate the basic results without forcing him to care about technical problems at a first reading. In Section 3 also the weak relative completion $\tilde{B}$ of B is introduced. It will turn out to be the largest space associated with B is a natural way. Furthermore, it will be proved that $\tilde{B}$ is always the dual of a Banach space in standard situation.

Section 4 represents the central part of this paper. There both kinds of module operations are considered simultaneously and their connections are clarified. Associated with each space B in standard situation there is a family of spaces, obtained by module theoretic methods. A number of formulas is combined to give Theorem 4.7, the main result of this paper. It contains a simple method of reduction (part A), and the "main diagram," showing that at most ten spaces can arise, which form two chains by inclusion, starting each with B_0 (closure of the test functions in B), and ending up with $\tilde{B}$. This diagram also visualizes the inherent symmetry between pointwise multiplication and convolution (in the Abelian case it can be made more concrete using the extended Fourier transforms). A good number of abstract and concrete results (e.g., concerning the Segal algebras of Reiter or the $A_p(G)$-algebras of Herz) can be obtained therefrom, some of them are formulated in Sections 5, 6.

In Section 5 various systematic results are stated. Among various possibilities we have tried to select those which are frequently required and useful for establishing the main diagram (and to draw informations from it) for concrete cases. Some of them, in particular Segal algebras, are discussed in Section 6, but the range of applications is much broader as will be explained in subsequent papers.

1. Banach Modules, Notations from Harmonic Analysis

As we shall treat Banach spaces having two module structures let us shortly recall some definitions and basic facts concerning Banach modules (cf. [7, 15, 25] for detailed expositions). A Banach space $(B, \| \|_B)$ is called a *left* (*right*) *Banach module* over a Banach algebra $(C, \| \|_C)$ (we write for short: B is a C-module), if it is a left (right) module over C, satisfying $\|cb\|_B \leqslant \|c\|_C \|b\|_B$ (or $\|bc\|_B \leqslant \|c\|_C \|b\|_B$) for all $c \in C$, $b \in B$. A closed subspace $D \subset B$ is called a *submodule* whenever $CD \subseteq D$. We shall be mainly interested in Banach algebras $(C, \| \|_C)$, having *bounded, two-sided, approximate units* (of norm $C_0 > 0$), i.e., for which there exists a bounded net $(u_\gamma)_{\gamma \in I}$ in C such that $\lim_{\gamma \to \infty} \|u_\gamma c - c\|_C = 0 = \lim_{\gamma \to \infty} \|cu_\gamma - c\|_C$ for all $c \in C$ (satisfying $\sup_\gamma \|u_\gamma\|_C \leqslant C_0$). There are two Banach spaces associated with a C-modul B in a natural way that we explain for left modules now: The *essential part of* B, written as B_C here, is defined as the closed linear span generated by the complex product CB. The *completion of* B (with respect to C) will be written as B^C, and is defined as the space of all *module homomorphisms from* C *to* B: $B^C := \{T \mid T: C \to B$, linear, bounded, $T(cc_1) = cT(c_1)$ for all c, $c_1 \in C\}$. Whenever B is *nondegenerate* as a C-module, i.e., whenever $cb = 0$ for all $c \in C$ implies $b = 0$, there is a natural injection $j_B : B \hookrightarrow B^C$, mapping b to T_b, given by $T_b(c) := cb$. Clearly this mapping is a continuous one, and allows us to consider B always as subspace of B^C. We call $(B, \| \|_B)$ a *strong C-module* whenever B is a closed subspace of B^C (i.e., for some equivalent norm it is strong in the sense of [15]). Finally we also mention that *the Banach dual* $(B', \| \|_{B'})$ of a left (right) C-module becomes a right (left) C-module by the transposed action, i.e., by the formula $\langle b'c, b\rangle = \langle b', cb\rangle$ $(\langle b, cb'\rangle = \langle bc, b'\rangle)$, $b \in B$, $b' \in B'$, $c \in C$. $(B', \| \|_{B'})$, endowed with this module structure will be called the *dual C-module* for B. It is a basic result due to Rieffel, that a dual Banach module is always complete, and that a reflexive Banach space is complete as well as essential (we assume that C has bounded approximate units [25, Theorem 8.9, Corollary 8.10]).

That the (convenient) convention of writing bc for the "product" in case of a right module may come in conflict with other conventions (which might be stronger, cf. below). Whether B as to be called a left or right C-module only depends on the corresponding law of associativity. In particular, any left module over a B^*-algebra $(C, \| \|_C)$ with involution (satisfying $c_1^* c_2^* = (c_2 c_1)^*$ for $c_1, c_2 \in C$) may also be considered as a right C-module by the action $(b, c) \to c^* b$, and vice versa.

Sometimes it will be convenient to consider locally convex algebras and locally convex modules, in which case the norm continuity simply has to be replaced by ordinary continuity of the module operations with respect to the given topologies.

For the *unit ball* $\{b \mid b \in B, \|b\|_B \leqslant 1\}$ in a Banach space $(B, \| \|_B)$ we use the symbol $\circ B$. *Continuous injections* between locally convex topological vector spaces will be written as $B^1 \hookrightarrow B^2$.

The relevant results concerning Banach modules and "elementary" formulas to be used extensively below are now collected.

THEOREM 1.1. *Let* $(C, \| \|_C)$ *be a Banach algebra having a bounded two-sided approximate unit* $(u_\gamma)_{\gamma \in I}$, *and let* $(B, \| \|_B)$ *be a left Banach module over* C. *Then one has*

(A) *The submodule B_C coincides with $\{b \mid b \in B, \lim_{\gamma\to\infty} \|u_\gamma b - b\|_B = 0\}$. Furthermore, the following factorization theorem holds true*: $B_C = CB$, *or more precisely, for $b \in B_C$ and $\varepsilon > 0$ there exists $b' \in B$ and $c \in C$ such that $b = cb'$, $\|b - b'\|_B < \varepsilon$ and $\|c\|_C \leqslant \sup_\gamma \|u_\gamma\|_C =: C_I$.*

(B) *Every essential submodule of B is contained in B_C, and any module homomorphism $T: B_C \to B^2$ satisfies $T(B_C) \subseteq B^2_C$.*

(C) *B^C is a C-module by the action $(cT)(c_1) = T(c_1 c)$, and $j_B : B \to B^C$ is a C-module homomorphism. Furthermore, $B \cong j_B(B)$ is dense in B^C for the strict topology.*

(D) *The following "elementary" formulas hold true*:

$$B_C = B_{CC} = {B^C}_C, \tag{1.1a}$$

$$B^C = B^{CC} = {B_C}^C. \tag{1.1b}$$

Proof. (A) This is [25, Proposition 3.4]. For a very short proof of the factorization theorem we refer to [13]. (B) is folklore and also clear from (A) ([25], Corollary 3.8]). The verification of (C) is not difficult (cf. [15]). Concerning (1.1a) we observe that $B_C = B_{CC}$ follows from the easy part of (A), and that (B), applied to $T = j_B$ gives $B_C \subseteq {B^C}_C$. The converse follows from the identity $cT(c_1) = T(c_1 c) = c_1\, Tc = c_1 b$, for $b := Tc$, showing that cT can be "represented" by multiplying with b from the right ($j_B(b) = cT$). The identity $B^C = {B_C}^C$ follows again from (B), using the fact that $C = C_C$. Combining these formulas one also obtains $B^{CC} = (B^C)_C{}^C = ({B^C}_C)^C = {B_C}^C = B^C$.

Concerning equivalent or at least sufficient conditions for B to be a strong C-module we have for C as above

LEMMA 1.2. (a) *$(B, \|\;\|_B)$ is a strong module if and only if $b \to \sup_\gamma \|u_\gamma b\|_B$ defines an equivalent norm on B. In particular, any essential module is strong.*

(b) *Let B be a closed subspace of B_1', where B_1 is an essential right Banach module over C. Then B is strong.*

Proof. (a) It is sufficient to show that the norms $\|\;\|_{BC}$ and as described above are equivalent on $B = j(B)$. We have $\|b\|_{BC} \geqslant (\sup_\gamma \|u_\gamma b\|)\, C_I^{-1}$, but on the other hand $\|b\|_{BC} = \sup_{c \in \circ C} \|cb\|_B = \sup_{c \in \circ C} \lim_\gamma \|cu_\gamma b\|_B \leqslant \sup_{c \in \circ C} \sup_\gamma \|c\|_C \|u_\gamma b\|_B$. For essential modules B we obtain directly

$$\|b\|_B = \lim_\gamma \|u_\gamma b\|_B \leqslant (\sup_\gamma \|u_\gamma\|_C)\, \|b\|_B \qquad \text{for all} \quad b \in B.$$

(b) The relevant estimate (showing that B_1' is a strong C-module) will be $\|b'\|_{B_1'} \geqslant \sup_\gamma \|u_\gamma b'\|_{B_1'}$ for $b' \in B_1'$. Given $\varepsilon > 0$ there exists $b \in \circ B_1$ such that $|\langle b', b\rangle| \geqslant \|b'\|_{B_1'} - \varepsilon$, and $\gamma_0 \in I$ such that $\|b - bu_\gamma\|_B < \varepsilon/\|b'\|_{B_1'}$ for all $\gamma \geqslant \gamma_0$. Combining these facts one obtains for $\gamma \geqslant \gamma_0$

$$\begin{aligned}\|u_\gamma b'\|_{B_1'} &\geqslant |\langle u_\gamma b', b\rangle| = |\langle b', bu_\gamma\rangle| \\ &\geqslant |\langle b', b\rangle| - |\langle b', bu_\gamma - b\rangle| \\ &\geqslant \|b'\|_{B_1'} - \varepsilon - \|b'\|_{B_1'} \|bu_\gamma - b\|_B \geqslant \|b'\|_{B_1'} - 2\varepsilon.\end{aligned}$$

In fact, we have thus shown that j_{B_1} is an isometry in this case.

In order to describe the spaces in "standard situation" below let us now fix some notations from harmonic analysis (cf. [23]). In the sequel G denotes a locally compact topological group with *Haar measure* dx, and *Haar modul* Δ. The symbols $(L^p(G), \| \|_p)$, $1 \leqslant p \leqslant \infty$ denote the usual Banach spaces of (equivalence classes of) measurable and p-integrable functions on G. $\mathscr{K}(G)$, the space of all continuous functions with compact support, is a dense subspace of $L^p(G)$ for $1 \leqslant p < \infty$, and the closure of $\mathscr{K}(G)$ in $L^\infty(G)$ is identified with $C^0(G)$. The space of continuous functions on G is written as $C(G)$, and $C_K := \{f \mid f \in C(G), \operatorname{supp} f \subseteq K\}$. $(L^1(G), \| \|_1)$ is considered as a *Banach convolution algebra*, i.e., as an algebra with respect to convolution, and $f \mapsto f^\sim : f^\sim(x) = \Delta^{-1}(x) f(x^{-1})$ defines an isometric involution of this Banach algebra. The *left* and *right translation* operators are given by

$$L_y f(x) = f(y^{-1}x), \qquad R_y f(x) = \Delta^{-1}(y) f(xy^{-1}), \qquad x, y \in G. \quad (1.2)$$

Sometimes we shall use the notations $f_y(x) := f(xy)$ and $f^\vee(x) := f(x^{-1})$. One has $\|L_y f\|_1 = \|f\|_1 = \|R_y f\|_1$ for all $y \in G$, $f \in L^1(G)$, and $\|L_y f\|_p = \|f\|_p$ for $f \in L^p(G)$, $y \in G$. Furthermore, one has continuity of $y \mapsto L_y f$ (or $R_y f$) from G into $L^p(G)$ for any $f \in L^p(G)$, $1 \leqslant p < \infty$.

For locally bounded and locally integrable function w on G satisfying $w(x) \geqslant 1$ and $w(xy) \leqslant w(x)\, w(y)$ for $x, y \in G$ is called a *weight function on* G (cf. [23, Chap. III, Sect. 7). If $w(x) = w(x^{-1})$ for all $x \in G$ the weight is called *symmetric* (we shall only use symmetric weights). The corresponding weighted L^1-space $L^1_w(G) := \{f \mid fw \in L^1(G)\}$ is a *Banach convolution algebra*, with the norm $\|f\|_{1,w} := \|fw\|_1$, called *Beurling algebra*. If w is symmetric, then L^1_w is also an involutive Banach algebra with respect to $f \to f^\sim$. It is well known (cf. [23]) that a Beurling algebra always has a bounded two-sided approximate unit $(e_\alpha)_{\alpha \in I}$ (we shall denote its norm by C_w). Observe that $(\tilde{e}_\alpha)_{\alpha \in I}$ is then a bounded two-sided approximate unit as well. Further terminology, in particular concerning *Segal algebras*, is taken from Reiter's books [23, 24]).

2. Test Functions, Distributions, Regularization

We come now to the description of a family of spaces of test functions on locally compact groups, and the corresponding spaces of distributions, i.e., their duals. Without making use of structure theory it is still general enough to serve as a useful tool for the treatment of Banach spaces "living" on general groups. It will be convenient to take up the convention that the symbol A will only be used to denote a "*nice*" pointwise Banach algebra on G, or more precisely, a Banach space $(A, \| \|_A)$ having the properties

(A1) $(A, \| \|_A)$ is continuously embedded in $(C^0(G), \| \|_\infty)$;

(A2) $(A, \| \|_A)$ is a regular, self-adjoint Banach algebra with respect to pointwise multiplication;

(A3) A is left and right invariant, i.e., $L_y A = A$ and $R_y A = A$ for all $y \in G$, and $y \to L_y f$ and $y \to R_y f$ are continuous mapping from G into $(A, \| \|_A)$, for any $f \in A$;

(A4) $A_0 := A \cap \mathscr{K}(G)$ is a dense subspace of $(A, \| \|_A)$;

(A5) $(A, \| \|_A)$ has bounded approximate units of norm $C_A > 0$.

Typical examples are $C^0(G)$, or the Fourier algebra $A(G)$ of an amenable group. Any homogeneous Banach space $(A, \| \|_A)$ which is at the same time a self-adjoint Wiener algebra in the sense of Reiter (cf. [23, Chap. II]) is a "nice" Banach algebra. Some informations concerning "nice" Banach algebras A are collected now for further reference.

Proposition 2.1. *Let $(A, \| \|_A)$ satisfy conditions* (A1)–(A3).

(A) *There exists a symmetric weight w on G such that A is an essential left as well as an essential right L^1_w-module with respect to ordinary left and right convolution.*

(B) *A satisfies* (A4) *and* (A5) *if and only if one of the following two conditions is satisfied*:

(A45) *A has bounded approximate units $(u_\gamma)_{\gamma \in L}$ in A_0.*

(ABT) *There is a bounded family $(\tau_\gamma)_{\gamma \in L}$ in A_0, which consist of "trapezoid" functions in the following sense: Given any compact set $K \subseteq G$ there exists $\gamma_0 \in M$ such that $\tau_\gamma(x) = 1$ on K for $\gamma \geqslant \gamma_0$.*

(C) *A_0 is a dense subspace of $\mathscr{K}(G)$, $C^0(G)$, and $L^1_w(G)$.*

(D) *Given any two bounded approximate units $(e_\alpha)_{\alpha \in I}$, $(e_\beta)_{\beta \in J}$ in $L^1_w(G)$ and $(u_\gamma)_{\gamma \in L}$ in A, $h \in A$ and $\varepsilon > 0$, it is possible to find $\alpha_0, \beta_0, \gamma_0$ such that one for $\alpha \geqslant \alpha_0$, $\beta \geqslant \beta_0$, $\gamma \geqslant \gamma_0$,*

$$\| e_\alpha * e_\beta * u_\gamma h - h \|_A < \varepsilon.$$

In particular, $\lim_\alpha \lim_\beta \lim_\gamma e_\alpha * e_\beta * u_\gamma h = h$ *in* A *(of course the limit may be taken in any other order as well).*

Proof. (A) It is obvious that $w(y) := \max(1, \|L_y\|, \|R_y\|, \|L_y^{-1}\|, \|R_y^{-1}\|)$ defines a submultiplicative and semicontinuous (hence measurable) function on G, with $w(y) = w(y^{-1})$ for all $y \in G$. By a Baire argument w has to be bounded on some open set, and therefore over compact sets, i.e., w is a symmetric weight. Observing further that the (pointwise defined) convolution products $k * h$ and $h * k$, $k \in L^1_w(G)$, $h \in A \subseteq C^0(G)$ may be interpreted as vector-valued integrals in A, given by

$$k * h = \int_G (L_y h)\, k(y)\, dy, \tag{2.1a}$$

$$h * k = \int_G (R_y h)\, k(y)\, dy \tag{2.1b}$$

assertion (A) is now clear.

(B) Assuming (A4) and (A5) a bounded approximate unit $(v_j)_{j \in J}$ in A_0 can be obtained by replacing its members (given by (A5)) term by term by (sufficiently close) elements from the dense subspace A_0. Renorming them if necessary one even may assume that they are of the same norm. That (A45) implies (A4) and (A5) is clear. In order to derive (ABT) therefrom observe the existence of a (possibly unbounded) family of trapezoid function $(w_l)_{l \in L}$ in A_0 follows from the existence of positive elements in A_0 (i.e. A2) and part (A). Using the method of [1] one verifies that a suitable subfamily of $(v_j + w_l - v_j w_l)_{(j,l) \in J \times L}$ defines a bounded approximate unit in A_0 (again it may be renormed to be of norm C_A). That any bounded system of trapezoid functions defines a bounded approximate unit for A follows from the fact that one has $\tau_\gamma h = h$ for $h \in A_K$, whenever $\gamma \leqslant \gamma_0$, again by approximation.

(C) Condition (A2) implies that A_0 contains nonzero, positive functions with arbitrarily small support near the origin. Renorming them in $L^1(G)$ one obtains a net $(g_\alpha)_{\alpha \in I}$ in A_0 satisfying $\|g_\alpha\|_1 = 1$ for all $\alpha \in I$, and $\|g_\alpha * k - k\|_\infty \to 0$ for $\alpha \to \infty$, for all $k \in \mathscr{K}(G)$, but $g_\alpha * k \in A_0 * \mathscr{K}(G) \subseteq A_0$. The family having common compact support convergence takes place in the topology of $\mathscr{K}(G)$. Density of $\mathscr{K}(G)$ in $L^1_w(G)$ and $C^0(G)$ gives the assertion.

(D) Let $h \in A$, $\varepsilon > 0$ be given. Assuming that the bounded approximate units are of norm C_I, C_J, C_L, respectively, it is possible to find α_0, β_0, and γ_0 such that one has for all $\alpha \geqslant \alpha_0$, $\beta \geqslant \beta_0$, and $\gamma \geqslant \gamma_0$, $\|e_\alpha h - h\|_A < \varepsilon/3$, $\|e_\beta h - h\|_A < \varepsilon/3C_I$, and $\|u_\gamma h - h\|_A < \varepsilon/3C_I C_J$. Then one has $\|e_\alpha * e_\beta * u_\gamma h - h\|_A \leqslant \|e_\alpha * e_\beta * (u_\gamma h - h)\|_A + \|e_\alpha * (e_\beta h - h)\|_A + \|e_\alpha h - h\|_A < \varepsilon$.

Remark 2.1. Since the spaces $\mathscr{K}(G)$, $C^0(G)$, and $L^1_w(G)$ (w symmetric) are $\check{}$-invariant the same arguments can be used to show density of $A_0^{\check{}}$ in these spaces. The same result is obtained using the fact that $(A^{\check{}}, \| \|_{A^{\check{}}})$, with $\|h^{\check{}}\|_{A^{\check{}}} := \|h\|_A$ as a norm, is again a nice algebra.

Given a nice Banach algebra A as above we consider $A_0 = A \cap \mathscr{K}(G)$ as a topological vector space, endowed with its natural inductive limit topology τ:

For a compact set $K \subseteq G$ the space $A_K := \{h \mid h \in A, \operatorname{supp} h \subseteq K\}$ is a Banach space with respect to $\| \|_A$ and A_0 is the inductive limit over a net of such spaces (the index set being a basis of compact sets ordered by inclusion). Thus, a net $(h_\alpha)_{\alpha \in I}$ is convergent to 0 in this topology if $\operatorname{supp} h_\alpha \subseteq K$, $\alpha \in I$, for some compact set $K \subseteq G$, and $\|h_\alpha\|_A \to 0$ for $\alpha \to \infty$. For $A = C^0(G)$ this gives the usual topology on $\mathscr{K}(G) = A_0$.

With this topology A_0 becomes a locally convex topological algebra, and even a topological module over A (hence ideal in A). It is also a left and right topological module with respect to ordinary convolution algebra $\mathscr{K}(G)$. For later use let us also mention that the right action of L^1_w on A, given by

$$h \cdot k := k^{\sim} * h = \int_G (L_{x^{-1}} h)\, k(x)\, dx \tag{2.2}$$

(it might be called *opposed left convolution*) when restricted to $\mathscr{K}(G)$ and A_0, respectively, gives A_0 a (different) *right* locally convex $\mathscr{K}(G)$-module structure.

It is clear from (2.1) that A_0 is a locally convex bimodule with respect to $*$. In general, however, the left action (2.1a) does not commute with the right action (2.2).

Our space of distributions will be of course A_0', the topological dual of A_0. There are at least two locally convex topologies of interest on A_0'. We shall mainly use the *weak topology* $\sigma(A_0', A_0)$ (and we write $\mu = \sigma\text{-}\lim_\alpha \mu_\alpha$ if μ_α is weakly convergent to μ in A_0'). The topology of uniform convergence on sets of the form $A_{K,n} := \{h \mid h \in A, \operatorname{supp} h \subseteq K, \ \|h\|_A \leqslant n\}$, $K \subseteq G$ compact, $n \in \mathbb{N}$ will be called *β-topology*. (This terminology coincides with the usual terminology for spaces of measures or distributions, where it is also called the strong topology; observe, however, that in the case of a general non-σ-compact space this is not the same as the strong topology in the sense of topological vector spaces (cf. [4, p. 64, Ex. 2, p. 65, Ex. 3, 8, p. 448, Remark b]).

The module structures on A_0 can be carried over to A_0' by transposition. Thus, A_0' becomes a (pointwise) topological A-module by the definition

$$\langle h\mu, g\rangle := \langle \mu, hg\rangle, \qquad \mu \in A_0', \quad g \in A_0, \quad h \in A. \tag{2.3}$$

Corresponding to the three $\mathscr{K}(G)$-module structures on A_0 given by (2.1a), (2.1b), and (2.2) there are the following two left and one right $\mathscr{K}(G)$-module structures on A_0',

$$\langle k * \mu, g\rangle := \langle \mu, k^{\sim} * g\rangle = \langle \mu, g \cdot k\rangle, \tag{2.2'}$$

$$\langle k \cdot \mu, g\rangle := \langle \mu, g * k\rangle, \qquad \mu \in A_0'. \tag{2.1b'}$$

$$\langle \mu \cdot k, g\rangle := \langle \mu, k * g\rangle, \qquad g \in A_0, \quad k \in \mathscr{K}(G), \tag{2.1a'}$$

We thus have again $\mu \cdot k = k^{\sim} * \mu$. Furthermore, it is clear that A_0' is a locally convex bimodule over $\mathscr{K}(G)$ with respect to $\cdot$. The justification for the use of the symbols $*$ and $\cdot$ (that have been already used for "ordinary" convolutions) is of course the fact that the new defined actions are natural extensions of the old ones. In other words, using essentially Fubini's theorem one shows that the natural embedding of $L^1(G) \supseteq L^1_w(G) \supseteq \mathscr{K}(G) \supseteq A_0$ and $C(G) \supseteq C_0(G) \supseteq A \supseteq A_0$ into A_0', given by

$$\langle f, g\rangle := \int_G f(x)\, g(x)\, dx, \qquad g \in A_0, \quad f \in L^1(G) \quad \text{or} \quad C(G),$$

is even a left $\mathscr{K}(G)$-module homomorphism if all spaces are endowed with their natural left $*$-structure. Of course a similar statement concerning $\cdot$ and pointwise products is valid.

Furthermore, we observe that the support of $\mu \in A_0'$ is defined in the usual way, and left and right translation operators are extended to A_0' by $\langle L_y\mu, g\rangle := \langle \mu, L_{y^{-1}}g\rangle$ and $\langle R_y\mu, g\rangle := \langle \mu, R_{y^{-1}}g\rangle$ for $y \in G$, $\mu \in A_0'$, $g \in A_0$.

Remark 2.2. Recall that the boundedness of $M \subseteq A_0'$ (with respect to the β-topology) can be characterized in the following way:

For every compact set $K \subseteq G$ there exists $C_K(M) > 0$ such that
$$|\langle \mu, g\rangle| \leqslant C_K(M)\, \|g\|_A \text{ for all } g \in A_k. \tag{2.3a}$$

Using the existence of trapezoid functions in A_0 it is not difficult to verify that this is equivalent to

$$hM \text{ is bounded in } (A', \|\ \|_{A'}) \text{ for each } h \in A_0 \tag{2.3b}$$

This is the same as equicontinuity in $\mathscr{L}(A_0, \mathbb{C}) = A_0'$ in the terminology of Schaefer (cf. [26, III.4.3]).

Now we have to introduce some operators which are compositions of multiplication and convolution operators. Let $k, k' \in \mathscr{K}(G)$ and $h \in A$ be given. Then the following (C = convolution, P = product) operators are well defined on A_0':

$$T: \mu \longmapsto k * (h\mu) = C_k P_h \mu \qquad \text{is a } CP\text{-operator,} \qquad (2.4a)$$

$$S: \mu \longmapsto h(k * \mu) = P_h C_k \mu \qquad \text{is a } PC\text{-operator,} \qquad (2.4b)$$

$$R: \mu \longmapsto k' * k * (h\mu) = C_{k'} C_k P_h \mu \qquad \text{is a CCP-operator.} \qquad (2.4c)$$

As will be seen immediately R stands for "regularization."

Convention. In order to simplify notations let us write $k * h\mu$ instead of $k * (h\mu)$ in the sequel. Thus in more general expressions of a similar form pointwise products are to be evaluated first, and convolutions then.

The basic properties of these operators, making them an extremely useful tool, are collected in the following results:

PROPOSITION 2.2. (A) *The operators T, S, and R introduced above are $\sigma - \sigma$-continuous linear operators on A_0'.*

(B) *If one has $k \in A_0^{\check{}} \cap C_{K'}$ and $h \in A_0 \cap C_{K''}$ (for suitable compact subsets K', $K'' \subseteq G$), then the operators T and S are σ-$\| \|_\infty$-continuous from A_0' to $C_{KK''}$ and $C_{K''}$, respectively, on any bounded subset M of A_0'.*

(C) *If furthermore $k' \in A_0 \cap C_{K'}$, then R is a σ-$\| \|_A$-continuous linear operator from any bounded subset $M \subseteq A_0'$ to $A_{K'KK''} \subseteq A_0$.*

Proof. (A) Since these operators are derived from continous operators on A_0 via transposition this part is clear.

(B) That the assumptions concerning A suffice in order to show that $A_0^{\check{}} * A_0' \subseteq C(G)$ has been shown in [12, Lemma 1.10], by using the formula $k * \mu(x) = \langle N_x k^{\check{}}, \mu \rangle$, where $N_z k(x) := \Delta(z) \Delta^{-1}(x) R_z k^{\check{}}(x)$, and the continuity of $z \mapsto N_z k$ as a mapping from G to A_0, for $k \in A_0$. That T maps A_0' into $C_{KK''}(G)$ follows now from the relation $\operatorname{supp}(k * h\mu) \subseteq (\operatorname{supp} k)(\operatorname{supp} h\mu) \subseteq KK''$. We know already from (A) that T is $\sigma - \sigma$-continuous, and the above formula shows that $\mu = \sigma\text{-}\lim_\alpha \mu_\alpha$ implies pointwise convergence of $T\mu_\alpha(x)$ to $T\mu(x)$ for all $x \in G$. If the net $(\mu_\alpha)_{\alpha \in I}$ belongs to M it is now also clear that $(k * h\mu_\alpha)_{\alpha \in I}$ is bounded and tight in $C^0(G)$. The equicontinuity of this net follows now by the estimate

$$|k * h\mu_\alpha(x) - k * h\mu_\alpha(y)| \leqslant \|N_x k - N_y k\|_A (\sup_{\mu \in M} \|h\mu\|_{A'}),$$

using the uniform continuity of $z \to N_z k$ over KK''. That weak convergence implies norm (i.e., uniform) convergence in that case follows from [12, Remark 2.1].

The local boundedness and locally uniform equicontinuity of the family $k * M$ suffices in order to show the corresponding result for S.

(C) One has $R\mu = k' * T\mu \in (A_0 \cap C_{K'}) * C_{KK''} \subseteq (A_0 * \mathscr{K}(G)) \cap$

$C_{K'} * C_{KK''} \subseteq A_0 \cap C_{K'KK''} = A_{K'KK''}$ for any $\mu \in A_0'$. Since weak convergence of a bounded net $(\mu_\alpha)_{\alpha \in I}$ in A_0' implies convergence of $T\mu_\alpha$ to $T\mu$ in $C_{KK''}$ by (B), we have convergence in $L^1_w(G)$, and by Proposition 2.1(A) convergence of $R\mu_\alpha$ to $R\mu$ in $(A, \| \|_A)$.

COROLLARY 2.3. *Let $(e_\alpha)_{\alpha \in I}$, $(e_\beta)_{\beta \in J}$, and $(u_\gamma)_{\gamma \in L}$ be nets in C_{K_1} and A_0, respectively (for some compact set $K_1 \subseteq G$), which are two-sided approximate units for $L^1_w(G)$ and A, respectively. Let $(R_\eta)_{\eta \in M}$, $R_\eta \mu := e_\alpha * e_\beta * u_\gamma \mu$, $M = I \times J \times L$ be a net of operators on A_0', M being ordered coordinatewise. Then one has $\mu = \sigma - \lim_\eta R_\eta \mu$ in A_0'. In particular, A_0 is σ-dense in A_0'.*

Proof. By the definition of these convolutions one has $\langle R_\eta \mu, g \rangle = \langle \mu, u_\gamma(\tilde{e}_\beta * \tilde{e}_\alpha * g) \rangle$ for all $g \in A_0$. Applying a variant of Proposition 2.1(D) and observing that the right term has common compact support, weak convergence of $R_\eta \mu$ in A_0' follows. The weak density of A_0 in A_0' is now an immediate consequence of Proposition 2.2(C), using the fact that the density of A_0 and $A_0^{\vee}$ in $L^1_w(G)$ (see Proposition 2.1(C) and Remark 2.1) and (A4), respectively, allow us to choose the approximate units in an appropriate way.

We conclude this section with a result concerning intersections of nice algebras (extending Proposition 2.1(C) that will be useful later (cf. Lemma 4.1).

LEMMA 2.4. *Let $(A^1, \| \|_{A^1})$ and $(A^2, \| \|_{A^2})$ be two Banach algebras satisfying* (A1)–(A4). *Then $(A^1 \cap A^2, \| \|_{A^1} + \| \|_{A^2})$ is a Banach algebra satisfying* (A1)–(A3), *which is dense in each of these algebras.*

Proof. Properties (A1) and (A3) are obvious. Concerning density we argue that $A_0^1 * A_0^2 \subseteq (\mathcal{K}(G) * A_0^2) \cap (A_0^1 * \mathcal{K}(G)) \subseteq A_0^1 \cap A_0^2$. Since $A^1 = A^1 * L^1_w(G)$ and $A^2 = L^1_w(G) * A^2$ (by the factorization theorem) the density of A_0^i in $L^1_w(G)$ (i.e., Proposition 2.1(C)) and the density of A_0^i in A^i imply the density of $A_0^1 * A_0^2$, hence $A^1 \cap A^2$, in both spaces. (A2) follows therefrom as well.

Remark 2.3. It has to be left as an open question whether $A^1 \cap A^2$ satisfies (A4), or even whether it has approximate units. In particular, we have not been able to show that the intersection of two "nice" algebras is again a "nice" algebra (except one assumes $A^1 A^2 \subseteq A^2$). It also seems to be difficult to find counterexamples. Of course, $(A^1 \cap A^2, \| \|_{A^1} + \| \|_{A^2})$ is a "nice" algebra if (and only if) there exists another nice algebra $(A^3, \| \|_{A^3})$, contained in $A^1 \cap A^2$ (use the system of trapezoid functions in A^3), which will be the case for all "practical" examples.

3. Banach Spaces in Standard Situation: Their Weak Relative Completion $\tilde{B}$

The situation described in Section 2 can be used as a general frame for the description of a class of Banach spaces having two module structures, at a fairly general level, and not only for spaces of locally integrable functions. The spaces considered will be called *spaces in standard situation.* Associated with each such space there is another, larger space $\tilde{B}$, called *the weak relative completion of B* (the use of the definite article will be justified later). It will be shown that usually $\tilde{B}$ can be characterized as the dual of another space in standard situation. The usefulness of *PC*- and *CP*-operators on such spaces will also become clear.

Definition 3.1. A Banach space $(B, \| \|_B)$ will be called to be in a *(left) standard situation with respect to a* (nice) *Banach algebra* $(A, \| \|_A)$ satisfying (A1)–(A5) (and the symmetric weight function w), if one has

(ST1) $A_0 \hookrightarrow (B, \| \|_B) \hookrightarrow A_0{}'$ is a chain of continuous injections;

(ST2) $(B, \| \|_B)$ is a Banach A-module with respect to pointwise multiplication (i.e., the above injections are A-module homomorphisms);

(ST3) $(B, \| \|_B)$ is a left Banach module over some Beurling algebra $L^1_w(G)$ with respect to convolution (i.e., B, considered on a left $\mathscr{K}(G)$-module, the above injections are $\mathscr{K}(G)$-module homomorphisms, the action on A_0 and $A_0{}'$ being left convolution.

It will be convenient to call (A, B) a left *standard pair* whenever the above conditions are satisfied. It is natural to call (A, B) an *essential standard pair* whenever B is an essential Banach module with respect to both actions.

The notion of a *right* standard situation (or a *right standard pair*) will be used in the case where one has $\cdot$ instead of $*$ in (ST3), i.e., left opposed convolution as a right $\mathscr{K}(G)$-action (cf. (2.2) and (2.1b$'$)).

Remark 3.1. Since one has $L^1_{w_1} \cap L^1_{w_2} = L^1_w$ (as Banach spaces), where $w(x) := \max(w_1(x), w_2(x))$ defines a weight function on G, it is no loss of generality to assume that the weight w arising in Section 2 is the same as that used in (ST 3).

Remark 3.2. As for $(A, \| \|_A)$ itself (cf. Proposition 2.1(A)), an essential left L^1_w-structure on B can be obtained whenever (ST1) is satisfied and B is left invariant with continuous shift, i.e., if $L_y B \subseteq B$ for all $y \in G$, and $y \mapsto L_y f$ is a continuous mapping from G into $(B, \| \|_B)$ for all $f \in B$. The converse is also true, as a consequence of the factorization theorem. If further A_0 is dense in B a compatible left L^1_w-module structure on B' can be obtained applying (2.2$'$).

Remark 3.3. For the continuity of the imbedding B in A_0' it is irrelevant whether one uses the σ- or the β-topology on A_0'. In fact, assuming $\| \, \|_B$-σ-continuity of the injection one can show that the mapping $L: b \mapsto L_b$, $L_b(g) = \langle b, g\rangle$, $g \in A_0$, defines a linear map from $(B, \| \, \|_B)$ to $(A_K, \| \, \|_A)$, having closed graph. The resulting continuity of L implies $\| \, \|_B$-β-continuity of the injection (cf. [12, Remark 2.2]).

Remark 3.4. Concerning left and right standard pairs it must be said that for symmetric weights (only such weights will be considered) a space B is in a left standard situation if and only if it is in a right standard situation. However, the kind of action of $\mathscr{K}(G)$ changes. Therefore we shall speak of a *standard pair* (A, B) whenever this aspect is without importance. Of course it would be possible (but without interest) to build up a symmetric theory for spaces B having a right convolution (as in (2.1b)) structure (hence a left structure by opposed right convolution).

Remark 3.5. It is important for the development of the theory that the "standard situation" described in [12] is slightly more general than that described here (cf. [12, Remark 1.2]). In particular, the results of that paper are available in our situation. The most relevant will be (cf. [12, Remark 2.1]): Given a bounded, right and (left) equicontinuous net $(f_\alpha)_{\alpha \in I}$ in $(B, \| \, \|_B)$ satisfying σ-$\lim_\alpha f_\alpha = \mu \in A_0'$, one has $\mu \in B$ and already norm convergence in $(B, \| \, \|_B)$. Recall that a bounded set M in $(B, \| \, \|_B)$ is *tight* if and only if for any $\varepsilon > 0$ there exists $g \in A_0$ such that $\| gf - f\|_A < \varepsilon$ for all $f \in M$. Replacing gf by $g * f$ gives a characterization of (left) *equicontinuity* of M in $(B, \| \, \|_B)$ (cf. [12, Propositions 2.4 and 2.5] for details).

Remark 3.6. Any two Banach spaces B^1 and B^2 which are in standard situation for A^1 and A^2, respectively, form a *compatible pair* (cf. [2, Chap. 2]). In fact, both are continuously embedded in $(A^1 \cap A^2)_0'$ (cf. Lemma 2.4). Consequently, inclusions between two such spaces are automatically continuous by the closed graph theorem. In particular, the norm on B is unique up to equivalence. Therefore B^1 is a closed subspace of B^2 if and only the norm $\| \, \|_{B^2}$, restricted to B^1, is equivalent to $\| \, \|_{B^1}$ on B^1.

DEFINITION 3.2. For a standard pair (A, B) we define $B_0 := \overline{A_0}^B$, and

$$\tilde{B} := \{\mu \mid \mu \in A_0', \mu = \sigma - \lim_\alpha f_\alpha \text{ for some bounded net } (f_\alpha)_{\alpha \in I} \text{ in } B\}, \tag{3.1}$$

and

$$\|\mu\|_\sim := \inf\{\sup_\alpha \|f_\alpha\|_B : \mu = \sigma - \lim_\alpha f_\alpha\}.$$

B is called the (weak) *relative completion of* B (in A_0'), and B_0 is called the (standard) *kernel of B*. We call (A, B_0) a *minimal*, and $(A, \tilde{B})$ a *maximal* standard pair.

Remark 3.7. As will be shown below the spaces B_0 and $\tilde{B}$ are actually independent of the weight function w and the "nice" Banach algebra A (combine Theorem 4.6 with Lemma 4.1).

LEMMA 3.1. *The spaces $(B_0, \| \|_B)$ and $(\tilde{B}, \| \|_\sim)$ are in standard situation with respect to A, and $B_0 \hookrightarrow B \hookrightarrow \tilde{B}$ is a chain of continuous injections.*

Proof. (i) It is clear that B_0 satisfies (ST1) as a closed subspace of $(B, \| \|_B)$. Since we have $A_0 A_0 \subseteq A_0$ and $A_0 * A_0 \subseteq A_0$ the continuity of these products and density of A_0 in B_0, A, and $L^1_w(G)$ imply conditions (ST2) and (ST3).

(ii) It is obvious that $\| \|_\sim$ defines a seminorm on $\tilde{B}$. Since (ST1) implies that for each compact $K \subseteq G$ there exists $C_K > 0$ such that $|\langle g, f_\alpha \rangle| \leqslant C_K \|f_\alpha\|_B \|g\|_A$ for all $g \in A_K$ and all $f_\alpha \in B$ it is clear by a limiting argument that $|\langle g, \mu \rangle| \leqslant C_K \|g\|_A \|\mu\|_\sim$ for $\mu \in \tilde{B}$. Therefore $\tilde{B}$ is continuously embedded in A_0' and $\| \|_\sim$ is a norm. The embeddings $B_0 \hookrightarrow B \hookrightarrow \tilde{B}$ are now obvious. The completeness of $(\tilde{B}, \| \|_\sim)$ can be shown as follows: Given any absolutely convergent series $\sum_{n=1}^\infty \|\mu_n\|_\sim =: C < \infty$ it is easily checked that $\mu := \sum_{n=1}^\infty \mu_n$ is convergent in (A_0', σ). Working with appropriate approximating nets in B it is a matter of routine to verify that $\mu \in \tilde{B}$, and that $\|\mu\|_\sim \leqslant C + \varepsilon$ for any given $\varepsilon > 0$, hence $\|\mu\|_\sim \leqslant C$. The completeness of $(\tilde{B}, \| \|_\sim)$ is then obvious.

Multiplication by elements of A as well as convolution by elements of $\mathscr{K}(G)$ being σ-continuous on A_0' one obtains the A-module structure (i.e., (ST2)) as well as the estimate $\|k * \mu\|_\sim \leqslant \|k\|_{1,w} \|\mu\|_\sim$ for $\mu \in \tilde{B}$ and $k \in \mathscr{K}(G)$. Applying the usual procedure it is possible to extend the action of $\mathscr{K}(G)$ on $\tilde{B}$ to an action of $L^1_w(G)$ on $\tilde{B}$, such that (ST3) is satisfied.

Attention. We shall use the symbol $k * \mu$ for $k \in L^1_w(G)$ and $\mu \in \tilde{B}$ in the above sense, even if it does not make sense to interpret it in the sense of Section 2. Since $L^1_w * A_0$ is not contained in A_0 it is also not reasonable to try to interpret $k * \mu$ as a weak limit of $k * f_\alpha$ for general $k \in L^1_w$ and $\mu \in \tilde{B}$.

Taking in account the above convention it is now possible to consider the *CP*- and *PC*-operators introduced in Section 2, acting on $\tilde{B}$, even for $k, k' \in L^1_w(G)$.

PROPOSITION 3.2. *Let $k, k' \in L^1_w(G)$ and $h \in A$ be given, and let T, S, R be the operators on $\tilde{B}$ given by $T\mu := k * h\mu$, $S\mu := h(k * \mu)$, and $R\mu := k' * k * h\mu$. Then one has*

(A) *T and S are compact operators from $\tilde{B}$ to B, satisfying*

$$\max(\|T\|, \|S\|) \leqslant \|k\|_{1,w} \|h\|_A. \tag{3.2}$$

(B) *R defines a compact linear operator from $\tilde{B}$ to B_0, satisfying*

$$\|R\| \leqslant \|k'\|_{1,w} \|k\|_{1,w} \|h\|_A. \tag{3.3}$$

Proof. Taking into account how the convolution operators are defined, and that A_0 as well as $A_0^{\vee}$ is a dense subspace of $L_w^1(G)$ it will be sufficient to verify the above assertions for k', $h \in A_0$ and $k \in A_0^{\vee}$ (in which case Proposition 2.2 is applicable).

(A) Let us show $T\tilde{B} \subseteq B$ (the result for S following by symmetric arguments). For $\mu \in \tilde{B}$ and $\varepsilon > 0$ there exists $(f_\alpha)_{\alpha \in I}$ in B satisfying $\mu = \sigma - \lim_\alpha f_\alpha$ and $\sup_\alpha \|f_\alpha\|_B \leqslant \|\mu\|_\sim + \varepsilon$. Then $(Tf_\alpha)_{\alpha \in I}$ is a bounded net in $(B, \| \|_B)$, such that

$$\sup_\alpha \|T_\alpha f\|_B \leqslant \|k\|_{1,w} \|h\|_A (\|\mu\|_\sim + \varepsilon),$$

which is tight and equicontinuous in $(B, \| \|_B)$, and σ-convergent to $T\mu$ by Proposition 2.2(A). By [12, Remark 2.1] we have $\lim_\alpha \|Tf_\alpha - T\mu\|_B = 0$. This implies $\|Tf\|_B \leqslant \sup_\alpha \|Tf_\alpha\|_B$ for each such net, or

$$\|Tf\|_B \leqslant \|k\|_{1,w} \|h\|_A \|\mu\|_\sim.$$

The compactness of T follows from [12, Theorem 2.1].

(B) The inclusion $R\tilde{B} \subseteq B_0$ is now an immediate consequence of Proposition 2.2(C). Since $R = k' * T$ its compactness follows from (A).

Formula (3.5a) is essentially a corollary to Proposition 3.2(B).

PROPOSITION 3.3. *The following spaces coincide as Banach spaces*

$$(B_0)_0 = B_0, \tag{3.4a}$$

$$(\tilde{B})^{\sim} = \tilde{B}, \tag{3.4b}$$

$$(B_0)^{\sim} = \tilde{B}, \tag{3.5a}$$

$$(\tilde{B})_0 = B_0. \tag{3.5b}$$

In particular, the norms of B and $\tilde{B}$ are equivalent on A_0.

Proof. Equation (3.4a) is obvious, and the proof of (3.4b) is left to the reader. One inclusion for (3.5a) is obvious, and in view of (3.4b) the inclusion $\tilde{B} \subseteq (B_0)^{\sim}$ follows from $B \hookrightarrow (B_0)^{\sim}$. Since $f = \sigma\text{-lim}_{\alpha,\beta,\gamma} e_\alpha * e_\beta * u_\gamma f$, Corollary 2.4 implies

$$\|f\|_{(B_0)^{\sim}} \leqslant (\sup_{\alpha,\beta} \|e_\alpha\|_{1,w} \|e_\beta\|_{1,w})(\sup_\gamma \|u_\gamma\|_A) \|f\|_B \leqslant C_w^2 C_A \|f\|_B.$$

In view of Banach's theorem, (3.5b) holds true if an only if $\|f\|_B \leqslant C \|f\|_{\sim}$ for $f \in A_0$ (the converse being obvious). Combining Corollary 2.4 with (3.2) this inequality can be derived, completing the proof of Proposition 3.3.

Next we give two useful (cf. Section 5) characterizations of B_0 and $\tilde{B}$ as subspaces of A_0', respectively.

LEMMA 3.4. *Let $(R_\eta)_{\eta \in M}$, $M = \{\eta := (\alpha, \beta, \gamma); \alpha \in I, \beta \in J, \gamma \in L\}$ be a net of operators on A_0' as considered in Corollary 2.3. Then one has*

$$B_0 = \{\mu \mid \mu \in A_0', (R_\eta \mu)_{\eta \in M} \text{ is a bounded Cauchy net in } (B, \|\ \|_B)\}; \tag{3.6a}$$

$$\tilde{B} = \{\mu \mid \mu \in A_0', (R_\eta \mu)_{\eta \in M} \text{ is a bounded family in } (B, \|\ \|_B)\}, \tag{3.6b}$$

in particular, one has

$$B_0 = \{\mu \mid \mu \in \tilde{B}, \mu = \lim_{\eta \to \infty} R_\eta \mu \text{ in } (\tilde{B}, \|\ \|_{\sim})\} \tag{3.6c}$$

whenever B is a closed subspace of $\tilde{B}$.

Proof. (a) The inclusion "$\subseteq$" is shown as in the case $B = A$ (cf. Proposition 2.1(D)). Conversely, any Cauchy net is convergent in B, and the limit must be μ by Corollary 2.3. But $R_\eta \mu \in B_0$ by Proposition 3.2(B), which implies "$\supseteq$", since B_0 is closed in $(B, \|\ \|_B)$.

(b) We always have $\mu = \sigma\text{-lim}_{\eta \to \infty} R_\eta \mu$ (Corollary 2.3). Therefore $\mu \in \tilde{B}$ if $R_\eta \mu$ is bounded in B. The converse follows from Proposition 3.2(B).

(c) Since $R_\eta \mu \in A_0 \subseteq B \subseteq \tilde{B}$ for all $\mu \in \tilde{B}$, and since the norms $\|\ \|_B$ and $\|\ \|_{\sim}$ are equivalent on the Banach space $(B, \|\ \|_B)$, Cauchy nets in B are just the convergent nets in $\tilde{B}$, and their limit is known to be μ.

Remark 3.8. Using the same arguments it can be shown that the existence of one of the iterated limits, such as $\lim_\alpha \lim_\beta \lim_\gamma e_\alpha * e_\beta * u_\gamma \mu$ in B already implies $\mu \in B_0$. In particular the limits (and also the operators) may be interchanged without effecting (the existence of) the limit in that situation.

In the last part of this section we show that $\tilde{B}$ is always the dual of a space in standard situation. A concrete description of the predual of $\tilde{B}$ is given as well.

For a better understanding of the arguments it will be useful to recall that one always has a continuous embedding of A_0 into B', whenever B satisfies (ST1). (Thus we may take $\tilde{B}$ as well.) In fact, considering $g \in A_0$ as a functional on A_0' we may restrict it to the subspace B. Identifying A_0 as a subspace of A_0' one can show that the injection is in fact continuous (cf. Remark 3.3).

THEOREM 3.5. *Let (A, B) be a standard pair such that B is closed in $\tilde{B}$. Then $\tilde{B}$ is a dual space. Furthermore*

$$\tilde{B} \cong (\overline{A_0}^{\tilde{B}'})', \tag{3.7}$$

the isomorphism being understood in the sense of Banach spaces.

Proof. Considered as a subspace of $\tilde{B}'$ the space A_0 separates points in $\tilde{B}$. Therefore, using a theorem due to Kaijser [16], we only have to show that the closed unit ball $\circ\tilde{B}$ of $\tilde{B}$ is compact in the $\sigma(\tilde{B}, A_0)$-topology. Since we know already (cf. the last part of the proof of Lemma 3.1) that $\circ\tilde{B}$ is equicontinuous in A_0' we may apply the Alaoglu–Bourbaki theorem (cf. [26, III.4.3]) in order to select from any given net in $\tilde{B}$ a σ-convergent subset, satisfying $\mu = \sigma\text{-}\lim_\alpha \mu_\alpha$ for some $\mu \in A_0'$. Since for any given $\varepsilon > 0$, each $\mu_\alpha \in \circ\tilde{B}$ can be obtained as a weak limit of a net $(f_\alpha^\beta)_{\beta \in J(\alpha)}$ in $(1+\varepsilon)\circ B$ one can obtain a net $(f_\gamma)_{\gamma \in L}$ in $(1+\varepsilon)\circ B$ such that $\mu = \sigma\text{-}\lim_\gamma f_\gamma$. Hence $\mu \in \tilde{B}$ and $\|\mu\|_\sim \leqslant 1+\varepsilon$ for each $\varepsilon > 0$, and the proof is complete.

Remark 3.9. The above theorem also holds true in more general situations. For example, it is sufficient to know that a Segal algebra S is closed in its vague relative completion $\tilde{S}$ (cf. [9]) in order to conclude that one has $\tilde{S} = (\overline{C_0}^{\tilde{S}'})'$. The proof is the same as above (cf. [5]).

Since the predual $\overline{A_0}^{\tilde{B}'}$ is not very convenient, the space $\tilde{B}'$ being usually not in standard situation (and also being difficult to describe) it is of interest to look for an (naturally) isomorphic standard space.

LEMMA 3.6. *Let (A, B) be a standard pair such that B is closed in $\tilde{B}$. Then $\overline{A_0}^{\tilde{B}'}$ is isomorphic to $\overline{A_0}^{B'}$ and to $\overline{A_0}^{B_0'}$ as a Banach space.*

Proof. Having the trivial isomorphism between A_0, considered as a subspace of $\tilde{B}'$ and A_0, considered as subspace of B' it will be sufficient to show that the corresponding norms on A_0 coincide (in order to establish the required isomorphism). We only have to verify $\| g\|_{\tilde{B}'} \leqslant \| g\|_{B'}$ for all $g \in A_0$.

Since, given $\varepsilon > 0$, any $\mu \in \circ\tilde{B}$ can be represented as the weak limit of a net $(f_\alpha)_{\alpha \in I}$ in $(1+\varepsilon)\circ B$ we have

$$|\langle g, \mu \rangle| = \lim_\alpha |\langle g, f_\alpha \rangle| \leqslant \|g\|_{B'}(1+\varepsilon)$$

for all $\mu \in \circ B$ and $\varepsilon > 0$. This gives the required estimate. Applying this argument to B_0 instead of B the second isomorphism is obtained.

In order to give a more precise description of the result concerning $\tilde{B}$ recall the connection between module structures and duality, in particular the fact that duality interchanges changes "left" and "right." The dual $(B', \| \|_{B'})$ of a space in standard situation is therefore a right module over A and $L^1_w(G)$. If, furthermore, A_0 is dense in B, it is not difficult to show that B' satisfies (ST1), and that the action of A and $L^1_w(G)$ obtained by transposition coincides with the ordinary action of A, and the action of $\mathscr{K}(G)$, restricted to B'. Starting with the right action $\cdot$ we would have obtained the left action $*$ of $\mathscr{K}(G)$ on B'. We thus have, using the terminology of Definition 3.1,

LEMMA 3.7. *Given a minimal left (right) standard pair, the pair (A, B') is a right (left) standard pair. This allows us to call (A, B_0') the* dual standard pair *to (A, B).*

Combining these results we obtain

THEOREM 3.8. *Given a (left) standard pair (A, B), the pair $(A, \tilde{B})$ is the dual pair of the (right) standard pair $(A, {B_0}'_0)$. In particular*

$$\tilde{B} \cong {B_0}'_0{}'. \tag{3.8}$$

Proof. Starting with formula (3.5a) we observe that it is sufficient to prove the result for minimal standard pairs, which are known to be closed in $\tilde{B}$ (cf. Proposition 3.3). Therefore Theorem 3.5 and Lemma 3.6 apply. The assertions concerning the module structures are a consequence of Lemma 3.7.

COROLLARY 3.9. *For a standard pair (A, B) one has*

$$(B_0')^{\sim} = B_0'. \tag{3.9}$$

Proof. Applying (3.5b) and (3.8) one has: $B_0' = B_0' = {B_0}'_0{}'_0{}' = (B_0')^{\sim}$.

Remark 3.10. This result, showing that the dual pair of a minimal is always maximal can also be seen as a consequence of Theorem 4.6, taking in account that a dual module is always complete.

COROLLARY 3.10. *Let (A, B) be a left standard pair. Then (A, B) is the dual pair of an essential right standard pair if and only if $B = \tilde{B}$.*

Proof. By (3.9) any dual space B_0' coincides with its completion. The converse is Theorem 3.8.

The embedding of A_0 into the dual B' of a space B in standard situation suggests to define the (combined) symbol B'_0 as to be $\overline{A_0}^{B'}$ in the above sense. Of course, this notation is consistent with that given in Definition 3.1 (i.e., $B'_0 = (B')_0$) whenever B' is in standard situation (cf. Remark 3.11). Applying this convention one obtains, in addition to formulas (3.4) and (3.5) the following results involving the tilde:

COROLLARY 3.11. *For a left standard pair (A, B) one has*

$$B_0'{}_0 \cong B'_0 \cong \tilde{B}'_0 \tag{3.10}$$

$$\tilde{B} = B_0'{}_0' = B'_0{}' = \tilde{B}'_0{}' \tag{3.11}$$

Proof. (3.10) follows from Lemma 3.6 using (3.5a). (3.8) combined with (3.10) gives (3.11).

COROLLARY 3.12. *A space B in standard situation is a reflexive Banach space if and only if one has $B_0 = B = \tilde{B}$, and $B' = B'_0$.*

Proof. (i) Assume the above conditions to be satisfied. Then one obtains, applying (3.8), $B = \tilde{B} = B_0'{}_0' = B'_0{}' = B''$ (the isomorphism corresponding to the natural one).

(ii) Now let a reflexive space B be given, i.e., with $B = B''$. Write $i: B'_0 \to B'$ for the natural injection. By checking that the transposed mapping $i'': B'' = B \to B'_0{}'$ is injective we may conclude that i has dense image, i.e., $B' = B'_0$. Thus, applying (3.10), B' may be identified with the standard space $B_0'{}_0$. This also gives $\tilde{B} = B_0'{}_0' = B'' = B$. Applying the above argument now to B' (instead of B) we obtain $B = B'' = B''_0 = B_0$, showing that all three conditions are satisfied.

Remark 3.11. This reasoning can be used to show that B' is in standard situation (in a natural way) if and only if $B = B'_0$.

Some applications of these results are to be discussed in the last section. We conclude this section with a short discussion of a certain property P (introduced by Krogstad for Segal algebras on Abelian groups, cf. [17]), trying to explain its role in this theory. Adapting the definition to our situation we say

DEFINITION 3.3. Let (A, B) be a standard pair. Then $(B, \| \|_B)$ has property P if there is a norm $\| \|$ on B equivalent to the given norm $\| \|_B$ such that

$$\|f\| := \sup\{|\langle f, g\rangle|, \|g\|_{(B,\| \|)'} \leqslant 1, g \in A_0\}, \qquad f \in B.$$

PROPOSITION 3.13. *Let (A, B) be a standard pair. Then B is closed in $\tilde{B}$ if and only if B satisfies property P.*

Proof. In view of Lemma 3.6 and the density of A_0 in B'_0 the above norm coincides with the norm of B as a subspace of $B_0{}'_0{}' = B'_0{}'$. However, by Theorem 3.5 this is just the norm $\| \|_\sim$, restricted to B. In view of Remark 3.6 the proof is complete.

Remark 3.12. This result is also true for symmetric Segal algebras which need not be in standard situation (cf. [5]). Using an example given in [9, Sect. 4, E] one obtains a Segal algebra without property P. This solves problems raised by Krogstad in [17]. The fact that a Segal algebra in standard situation always satisfies P explains perhaps why one could not find a counterexample among the Segal algebras considered usually.

4. ESSENTIAL PARTS, MODULE COMPLETIONS, AND THE MAIN DIAGRAM

Having introduced Banach spaces $(B, \| \|_B)$ in standard situation in Section 3, which have two module structures with respect to Banach algebras having bounded approximate units, it makes sense to consider the corresponding essential parts and completions as introduced in Sections 1 in the abstract setting. As we shall see these spaces are again Banach spaces in standard situation, which allows to iterate this procedure. As it will be shown, however, instead of an infinite family of spaces at most ten spaces arise this way, which are nicely related by chains of inclusions (explained by the main diagram).

Notation. Let $(B, \| \|_B)$ be in standard situation with respect to A and L^1_w. Then we write B_A and $B_G := B_{L^1_w} = L^1_w * B$ for the *essential parts*, and B^A and $B^G := B^{L^1_w}$ for the *module completions*, respectively. (Thus (A, B) is an essential standard pair whenever $B_A = B = B_G$.)

LEMMA 4.1. (a) *$h \in B_A$ if and only if h can be approximated in $(B, \| \|_B)$ be compactly supported elements.*

(b) *$h \in B_{L^1_w}$ if and only if $y \mapsto L_y h$ is a continuous map from G into $(B, \| \|_B)$.*

(c) *The spaces* B_A *and* B_G *are independent from the particular nice algebra* A *(satisfying* (A1)–(A5)) *as well as of the weight function* w.

(d) *The same can be said for* B^G *and* B^A.

Proof. We only discuss (d): For B^G cf. [12, Theorem 3.5] and for B^A one argues as follows: Let A^1 and A^2 be two nice Banach algebras, and let T be a bounded (pointwise) multiplier from A^1 to B. By Proposition 2.2(B) there exists a bounded approximate unit $(u_\gamma)_{\gamma\in I}$ of trapezoid functions in A^1 of norm C_1. Hence for each $g\in A_0^1\cap A_0^2$ there exists $u\in A^1$, satisfying $\| \|_{A^1}\leqslant C_1$ and $u(x)=1$ on supp g. This implies

$$\|T(g)\|_B=\|T(ug)\|_B=\|gTu\|_B\leqslant\|g\|_{A_2}\|T\|\,\|u\|_{A_1}\leqslant C\|g\|_{A_2}.$$

Since $A_0^1\cap A_0^2$ is a dense subspace of A^2 (cf. Lemma 2.4) it follows that T extends to a continuous multiplier from A^2 to B. The assertion now follows for reasons of symmetry.

The main result concerning "essential parts" can now be stated as

THEOREM 4.2. *Given a Banach space* $(B,\| \|_B)$ *in standard situation, the closed subspaces* $(B_A,\| \|_B)$ *and* $(B_G,\| \|_B)$ *are again in standard situation. Furthermore, one has*

$$B_{AG}=B_{GA}=B_A\cap B_G=B_o. \tag{4.1}$$

Proof. (i) At the first step we only have to check that there is an A-module structure on B_G and a left L_w^1-module structure on B_A. Using the continuity of translation in $(A,\| \|_A)$ it is easy to see $y\mapsto L_y(hf)$ is continuous from G to $(B,\| \|_B)$ for $f\in B_G$, i.e., that $AB_G\subseteq B_G$. Let now $k\in L_w^1(G)$ and $f\in B_A$ be given. Then one has $f=\lim_j u_jf$ in B for a suitable sequence $(u_j)_{j\in\mathbb{N}}$ in A_0 and $k=\lim_n k_n$ in $L_w^1(G)$ for a suitable sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathscr{K}(G)$. Since k_n*u_jf has compact support and since A contains trapezoid functions this implies $k_n*\ u_jf\in AB=B_A$ for $j,n\in\mathbb{N}$. B_A being closed in $(B,\| \|_B)$ it follows that $k*f\in B_A$, hence $L_w^1*B_A\subseteq B_A$.

(ii) In order to prove (4.1) we observe that the inclusions $A_0\subseteq B_{AG}\subseteq B_A\cap B_G$ and $A_0\subseteq B_{GA}\subseteq B_A\cap B_G$ are obvious. It therefore remains to show that A_0 is a dense subspace of $(B_A\cap B_G,\| \|_B)$. Since $B_A\cap B_G$ is an essential module with respect to both actions one shows (cf. the proof of Proposition 2.1(D)) that for $\varepsilon>0$ and $f\in B_G\cap B_A$ there exists a CCP-operator R as in 2.2(C) such that $\|Rf-f\|_B<\varepsilon$ for any given $f\in B_G\cap B_A$. But $Rf\in R(A_0)\subseteq A_0$, which completes the proof of (A) (4.1).

Remark 4.1. (A,B) is thus an *essential* standard pair if and only if A_0 is a dense subspace of $(B,\| \|_B)$. (cf. [12, Lemma 1.5]).

Remark 4.2. It follows now that the regularizing operators map B into B_0. Applying the factorization theorem twice one obtains: Given $f \in B$ it belongs to B_0 if and only if there exists $f' \in B$, $h \in A$, and $k \in L^1_w(G)$ such that $f = h(k * f')$, or equivalently $f = k * hf'$.

We want to show now that the Banach spaces B^A and B^G may be considered as Banach spaces in standard situation. In doing this we even obtain more general results.

LEMMA 4.3. *The Banach spaces B^A and B^G (each endowed with the operator norm) are continuously embedded in $\tilde{B}$. More precisely, one has*

$$B^A = \{\mu \mid \mu \in \tilde{B}: h\mu \in B \text{ for all } h \in A\}, \tag{4.2a}$$

$$B^G = \{\mu \mid \mu \in \tilde{B}: k * \mu \in B \text{ for all } k \in L^1_w\}, \tag{4.2b}$$

Proof. For $T \in B^A \subseteq H_A(A, B)$ consider the net $(T(u_\gamma))_{\gamma \in I}$, which is bounded in $(B, \|\ \|_B)$ whenever $(u_\gamma)_{\gamma \in I}$ is a bounded approximate unit in $(A, \|\ \|_A)$ of norm C_A. It follows that there exists a subset $(u_\beta)_{\beta \in J}$ and $\mu \in A_0'$ such that $\mu = \sigma\text{-}\lim_\beta T(u_\beta)$; hence $\mu \in \tilde{B}$, and

$$h\mu = \sigma\text{-}\lim_\beta hT(u_\beta) = \sigma\text{-}\lim_\beta T(u_\beta h) = Th \qquad \text{for} \quad h \in A.$$

Furthermore

$$\|\mu\|_\sim \leqslant \sup_\gamma \|T(u_\gamma)\|_B \leqslant C_A \|T\|.$$

Conversely, any $\mu \in A_0'$ satisfying $h\mu \in B$ defines an element of B^A by the closed graph theorem. The argument for B^G is essentially the same (observe that the expressions $h\mu$ and $k * \mu$ are always well defined).

Remark 4.3. Considering the chain $B \hookrightarrow B^A \hookrightarrow \tilde{B}$ it is now clear that B is a strong A-module if B is closed in $\tilde{B}$, the norms $\|\ \|_B$ and $\|\ \|_\sim$ (and hence $\|\ \|_{B^A}$) being equivalent on B. In a similar way one concludes that B is a strong L^1_w-module if B is closed in $\tilde{B}$ (cf. Corollary 4.9).

There is another characterization of B^A and B^G, respectively.

PROPOSITION 4.4. *Let $(e_\alpha)_{\alpha \in I}$ in $\mathscr{K}(G)$ and $(u_\gamma)_{\gamma \in L}$ in A be bounded two-sided approximate units for $L^1_w(G)$ and A, respectively. Assume that B is closed in $\tilde{B}$. Then one has*

$$B^A = \{\mu \mid \mu \in A_0', (u_\gamma \mu)_{\gamma \in L} \text{ is a bounded net in } B\} \tag{4.3a}$$

$$B^G = \{\mu \mid \mu \in A_0', (e_\alpha * \mu)_{\alpha \in I} \text{ is a bounded net in } B\}. \tag{4.3b}$$

Furthermore, the expressions $\sup_\gamma \|u_\gamma \mu\|_B$ and $\sup_\alpha \|e_\alpha * \mu\|_B$ define equivalent norms on these spaces.

Proof. The inclusion "$\subseteq$" as well as the corresponding estimate is obvious by Lemma 4.3. In order to show the converse for (4.3a) we have to show that $h\mu \in B$ for any $h \in A$. In fact, we have $hu_\gamma \mu \in B$ for each μ as above. Since $\lim_\gamma \|hu_\gamma - h\|_A = 0$ for any $h \in A$ the estimate $\|hu_\gamma \mu - hu_{\gamma'} \mu\|_\sim \leqslant \|hu_\gamma - hu_{\gamma'}\|_A \|\mu\|_\sim$ shows that $(hu_\gamma \mu)_{\gamma \in L}$ is a Cauchy net in $(\tilde{B}, \| \|_\sim)$, hence $(B, \| \|_B)$. Of course we have $h\mu = \lim_\gamma hu_\gamma \mu$ in B, and therefore $\|h\mu\|_B \leqslant \|h\|_A \sup_\gamma \|u_\gamma \mu\|_B$, giving the desired estimate. For (4.3b) a completely symmetric argument has to be given.

Remark 4.4. Since the closedness of B in $\tilde{B}$ implies that B is closed in B^A as well as in B^G (cf. Remark 4.3) it is of interest to note that it would have been sufficient to suppose for (4.3a) to be true that B is a strong A-module. In fact, one has to replace the first estimate above by

$$\begin{aligned} \|h\mu\|_B &\leqslant C \sup_\gamma \|u_\gamma h\mu\|_B = C \sup_\gamma \|hu_\gamma \mu\|_B \\ &\leqslant C \|h\|_A \sup_\gamma \|u_\gamma \mu\|_B \qquad \text{for} \quad h \in A. \end{aligned}$$

Since we have used $u_\gamma h = hu_\gamma$ for this argument the corresponding version for (4.3b) only applies if one has central approximate units in $L^1_w(G)$. Thus one can say that (4.3b) is also true for standard spaces B on [SIN]-groups (cf. [18, 22]) which are strong L^1_w-modules.

Remark 4.5. It is easy to derive from (4.3a), using trapezoid functions in A (cf. Proposition 2.1(B)) that one has $\mu \in B^A$ if and only if there exists $C > 0$ such that for any $K \subseteq G$, compact, one can find $f \in B$, $\|f\|_B \leqslant C$, such that $\langle \mu - f, g \rangle = 0$ for all $g \in A_K$ (i.e., $\mu = f$ on K).

Because we shall make use of the spaces $\tilde{B}_A$ and $\tilde{B}_G$ let us now give characterizations of these spaces as subspaces of $A_0{}'$ and show inclusions between them and the spaces considered above. Since $\tilde{B} = (B_0)^\sim$ by Corollary 3.3 it would be no loss of generality to assume that (A, B) is an essential standard pair.

LEMMA 4.5. *Let (A, B) be a standard pair. Then one has*

$$(\tilde{B})_A = \tilde{B}^{tt} := \{\mu \mid \mu \in A_0{}', \mu = \sigma\text{-}\lim_\alpha f_\alpha, \textit{for some bounded, tight net } (f_\alpha)_{\alpha \in I} \textit{ in } B\}; \tag{4.4a}$$

$$(\tilde{B})_G = \tilde{B}^{eq} := \{\mu \mid \mu \in A_0{}', \mu = \sigma\text{-}\lim_\alpha f_\alpha, \textit{for some bounded, equicontinuous net } (f_\alpha)_{\alpha \in I} \textit{ in } B\}, \tag{4.4b}$$

each of these spaces being endowed with its canonical norm, and

$$(\tilde{B})_A \cap (\tilde{B})_G = B_0. \tag{4.4c}$$

Proof. By the factorization theorem (Theorem 1.1(A)) each $\mu \in (\tilde{B})_A$ is of the form $\mu = h\mu'$, $\|h\|_A \leqslant C_A$, $\|\mu'\|_\sim \leqslant \|\mu\|_\sim + \varepsilon$. Therefore $\mu = \sigma\text{-lim}_\alpha\, hf'_\alpha$ (a bounded, tight net), for any bounded net satisfying $\mu' = \sigma\text{-lim}_\alpha\, f'_\alpha$. Since any tight net is of this form (cf. [12, Proposition 2.4]), the converse, as well as the corresponding norm estimates concerning (4.4a) follow. Again (4.4b) is proved by the same arguments, and (4.4c) follows from (4.1) and (3.5b).

Returning now to the general situation we have

PROPOSITION 4.5. *For a standard pair (A, B) the following inclusions holds true:*

$$(\tilde{B})_A \subseteq B^G, \tag{4.5a}$$

$$(\tilde{B})_G \subseteq B^A. \tag{4.5b}$$

Proof. We prove only the first formula. For $\mu' = h\mu \in (\tilde{B})_A = A \cdot \tilde{B}$ we have $\mu' = h\mu$ for some $h \in A$, $\mu \in \tilde{B}$. Therefore $k * \mu' = k * h\mu = T\mu \in B$ for each $k \in L^1_w(G)$, by Theorem 3.2, i.e., $\mu' \in B^G$ by Lemma 4.3.

THEOREM 4.6. *Given a Banach space $(B, \| \|_B)$ in standard situation, the spaces B^A and B^G, endowed with their natural norms are again in standard situation. Furthermore one has*

$$B^{AG} = \tilde{B} = B^{GA}. \tag{4.6}$$

Proof. Let us consider B^A and prove $B^{AG} = \tilde{B}$. ($B^{GA} = \tilde{B}$ following the same way). We know already (Lemma 4.3) that the embeddings $A_0 \hookrightarrow B \hookrightarrow B^A \hookrightarrow \tilde{B} \hookrightarrow A_0'$ are continuous, and that B^A is an A-module (Theorem 1.1(B)). Using (4.2a) and (4.5b) we obtain

$$L^1_w * B^A \subseteq L^1_w * \tilde{B} = (\tilde{B})_G \subseteq B^A$$

(together with the corresponding norm inequality).

Therefore B^A is in standard situation and $B^{AG} = (B^A)^G \subseteq \widetilde{B^A} \subseteq \tilde{\tilde{B}} = \tilde{B}$ and the easy part of (4.2). Using the other direction of Lemma 4.3 we see that the inclusion $\tilde{B} \subset B^{AG}$ follows if we know $h(k * \mu) \in B$ for each $h \in A$, $k \in L^1_w(G)k$, but this is just Theorem 3.2(B).

It is now clear that it is possible to define, by induction, spaces that can be described by a finite chain of upper or lower "exponents" A and G, such as ${B^A}_G{}^{AG}$. As its turns out, however, one fortunately does not obtain more than ten different spaces in this way. More precisely, we obtain the following main result of this paper:

THEOREM 4.7. (A) *Any symbol having a finite chain of upper or lower exponents can be reduced (i.e., the corresponding spaces coincide as Banach spaces) to a symbol carrying either one (upper or lower) or the two exponents A and G in suitable positions. The reduction of complicated symbols may be carried out by leaving each of the exponents (A and G) in its last position, while eliminating all preceding occurrences. In particular, one has*

$$(B_0)^G = B_A{}^G, \tag{4.7a}$$

$$(B_0)^A = B_G{}^A, \tag{4.7b}$$

$$(\tilde{B})_A = B^G{}_A, \tag{4.8a}$$

$$(\tilde{B})_G = B^A{}_G \tag{4.8b}$$

(B) *The* $10 = 4 + 6$ *spaces arising on the most general case form two chains, ordered by inclusion, that may be described by (the bigger spaces being above the smaller ones):*

$$\begin{tikzcd}[arrows=dash]
 & \tilde{B} \arrow[dl, "+1"'] \arrow[dr, "+2"] & \\
B^A \arrow[d, "+3"'] & & B^G \arrow[d, "+4"] \\
B_G{}^A \arrow[d, "5"'] & & B_A{}^G \arrow[d, "6"] \\
B^A{}_G \arrow[d, "-2"'] & & B^G{}_A \arrow[d, "-1"] \\
B_G \arrow[dr, "-4"'] & & B_A \arrow[dl, "-3"] \\
 & B_0 &
\end{tikzcd}$$

(C) *Each of these ten inclusions may be proper, as can be seen by considering either* $C^0(\mathbb{R})$ *or* $L^\infty(\mathbb{R})$.

(D) *The numbers* $\{+1, \ldots, -4, 5, 6\}$ *indicate that there are "coupled coincidences." Given a space* $(B, \|\ \|_B)$ *in standard situation the inclusions* *numbered by $+i$ becomes equality if and only if the inclusion numbered by $-i$, $1 \leqslant i \leqslant 4$, is an equality. (The coincidences 5 and 6 are "free").*

Proof. (A) It is easy to derive the indicated rule combining the "elementary formulas" (11) with (4.1) and (4.6). Since any double occurrence of one of the symbols G or A can be eliminated by this method only the ten spaces illustrated above remain.

(B) The chain $B_G \subseteq B \subseteq B^G$ implies (by taking A-essential parts) $B_0 = B_{GA} \subseteq B_A \subseteq {B^G}_A$ which describes the inclusions between the A-essential modules in the diagram. In a similar way the inclusions in each quarter of the diagram are obtained. Since we have $(\tilde{B})_G = {B^{AG}}_G = {B^A}_G$ the remaining two (nontrivial) inclusions (numbered 5 and 6) are just given by (4.5).

(C) The spaces $C^0(G)$ and $L^\infty(G)$ will be discussed in Examples 6.2 and 6.3.

(D) Applying the reduction method of (A) it is easy to verify the "coupled coincidences." For example,

$$+1 \Leftrightarrow -1: \quad B^A = \tilde{B} = B^{GA} \Rightarrow B_A = {B^A}_A = (B^{GA})_A = {B^G}_A$$
$$\Rightarrow B^A = {B_A}^A = ({B^G}_A)^A = B^{GA} = \tilde{B}.$$

Coincidences 5 and 6 only imply tautologies by such methods.

Remark 4.6. The above diagram will be an important and useful tool for "seeing" and proving the results given in the sequel. It will be helpful for the reader to visualize statements of Section 5 by drawing the corresponding reduced diagrams.

We conclude this section with a short discussion of the question, under what conditions B is closed in $\tilde{B}$ (cf. above).

LEMMA 4.8. (a) *If B is closed in B^A, then B^G is closed in $\tilde{B}(= B^{GA})$.*

(b) *If B is closed in B^G, then B^A is closed in $\tilde{B}(= B^{AG})$.*

Proof. We only prove (a). Let $\mu \in B^G$ be given. Then for $\varepsilon > 0$ there exists $k \in \circ L^1_w(G)$ such that $\|\mu\|_{BG} \leqslant \|k * \mu\|_B + \varepsilon$, by the definition of $\| \, \|_{BG}$. Assuming that $\|f\|_B \leqslant C \|f\|_{BA}$ for some $C > 0$ and all $f \in B$ we can find $h \in A$ with $\|h\|_A \leqslant 1$ such that

$$\|k * \mu\|_B \leqslant C(\|h(k * \mu)\|_B + \varepsilon).$$

Now applying (3.2) we obtain

$$\|\mu\|_{BG} \leqslant C \|h(k * \mu)\|_B + \varepsilon(C + 1) \leqslant C \|h\|_A \|k\|_{1,w} \|\mu\|_\sim + \varepsilon(C + 1).$$

Since $\varepsilon > 0$ has been arbitrary we have $\|\mu\|_{BG} \leqslant C \|\mu\|_\sim$.

COROLLARY 4.9. *B is closed in $\tilde{B}$ if and only if B is a strong A-module as well as a strong L^1_w-module.*

Proof. Assume that the norms of B and $\tilde{B}$ are equivalent on B. Then the norms of B^A and B^G have to be equivalent to $\| \|_B$ on B by Lemma 4.3. Conversely, one has $B \subset B^A$ and by Lemma 4.8(a) $B^A \subset \tilde{B}$ as closed subspaces, respectively , which together implies that B is closed in $\tilde{B}$.

Remark 4.7. Once more applying Lemma 4.8 we see that B^A and B^G are closed in $\tilde{B}$ whenever B is closed in $\tilde{B}$.

Remark 4.8. Checking the proofs once more it is easy to see that all embeddings are isometric ones whenever A and $L^1_w(G)$ have approximate units of norm 1 and $B \hookrightarrow B^A$ and $B \hookrightarrow B^G$ are isometries (which correspond to the situation occurring "in practice").

5. SYSTEMATIC RESULTS

It is the purpose of this section to present some systematic results. In the first part we collect virtually all results that can be obtained from Theorem 4.7 by purely formal operations. Certain particular cases that are typical for situations arising naturally are discussed a little bit more carefully. Then some formulas concerning sums and intersections of spaces in standard situation, and concerning their duality are given.

In order to keep the description of our results now as short as possible, we take up the *convention*:

Given a space B in standard situation write $E(B)$ *for the subset of* $\{1, 2,..., 6\}$ *corresponding to the equalities*. We shall also say that B satisfies E_i (E for equality) of (and only if) $i \in E(B)$. Thus, for example $E(B) = \{1, 2, 3, 4\}$ indicates that the diagram reduces to exactly two different spaces $\tilde{B}$ and B_0 (otherwise, E_5 and E_6 would both be true). Any (A- or G-) complete module coincides with $\tilde{B}$, and any essential module in the diagram coincides with B_0 in that case (cf. (5.9) in Table I).

The utility of the main diagram will become clear to the reader by drawing diagrams corresponding to the situations discussed below.

PROPOSITION 5.1. *For any standard pair (A, B) any two assertions belonging to the same block of Table I are equivalent. This table has to be interpreted in the following sense: Any two conditions to be found in the same column of the above table are equivalent. Furthermore one can say, for example, that each of the equalities ${B^G}_A = B_G$ and $B^G = {B_G}^A$ imply the properties listed in* (5.5) *(i.e., $\{1, 3\} \subseteq E(B)$), and similarly ${B^A}_G = B_A$ or $B^A = {B_A}^G$ imply $\{2, 4\} \subseteq E(B)$ (cf.* (5.6)).

TABLE I

(5.1) $1 \in E(B)$ $B^A = \tilde{B}$ $B_A = B^G{}_A$ $B^G \subseteq B^A$ $B_A{}^G \subseteq B^A$ $B^G{}_A \subseteq B^A$	(5.2) $2 \in E(B)$ $B^G = \tilde{B}$ $B_G = B^A{}_G$ $B^A \subseteq B^G$ $B_G{}^A \subseteq B^G$ $B^A{}_G \subseteq B^G$	(5.3) $3 \in E(B)$ $B_A = B_0$ $B^A = B_G{}^A$ $B_A \subseteq B_G$ $B_A \subseteq B^A{}_G$ $B_A \subseteq B_G{}^A$
(5.4) $4 \in E(B)$ $B_G = B_0$ $B^G = B_A{}^G$ $B_G \subseteq B_A$ $B_G \subseteq B^G{}_A$ $B_G \subseteq B_A{}^G$	(5.5) $\{1, 3\} \subseteq EB$ $B_G{}^A = \tilde{B}$ $B^G{}_A = B_0$ $B^G{}_A \subseteq B^A{}_G$ $B^G{}_A \subseteq B_G{}^A$ $B_A{}^G \subseteq B_G{}^A$	(5.6) $\{2, 4\} \in EB$ $B_A{}^G = \tilde{B}$ $B^A{}_G = B_0$ $B^A{}_G \subseteq B^G{}_A$ $B^A{}_G \subseteq B_A{}^G$ $B_G{}^A \subseteq B_A{}^G$
(5.7) $\{1, 2\} \in R(B)$ $B^A = B^G$	(5.8) $\{3, 4\} \in E(B)$ $B_A = B_G$	(5.9) $\{1, 2, 3, 4\} \subseteq E(B)$ $B^A{}_G = B^G{}_A$ $B_A{}^G = B_G{}^A$

The proofs of these assertion can left to the reder since they are completely elementary: Starting from one of the inclusions one obtains the other ones by (iterated) taking essential parts or completions with respect to $L^1_w(G)$ or A.

Remark 5.1. Observe that Table I shows complete symmetry between "A" and "G", i.e., formally (5.3) can be obtained by interchanging the symbols G and A in (5.1). The same relations holds between (5.2) and (5.4), and (5.5) and (5.6). It is also possible to "obtain" (5.2) from (5.1) and (5.4) from (5.3) by leaving the symbols in their order, but interchanging their positions (up $\leftrightarrow$ down) as well as the order of inclusions. Of course, this symmetry of tables is a consequence of the symmetry between convolution and multiplication that become appearent already in Section 4. Any result that can be interpreted in the diagram for certain spaces in standard situation has, so to speak, a dual companion, which is obtained by interchanging the roles of A and G. In practice it allows us to point out connections between different results in the literature, but also to establish results concerning (Fourier) multipliers, starting from a result on pointwise products, by dualization (of the result and its proof).

TABLE II

(5.10)

$\cap$	B_G	$B^A{}_G$	$B_G{}^A$	B^A
B_A	B_0	B_0	B_0	B_A
$B^G{}_A$	B_0	B_0	B_0	B_A
$B_A{}^G$	B_0	B_0	**	**
B^G	B_G	B_G	**	**

It is our next aim to describe the spaces that can be obtained by taking intersections of spaces appearing in the diagram. Of course, one obtains closed subspaces of $\tilde{B}$ whenever B is closed in $\tilde{B}$, but it is surprising that most spaces arising this way coincide with spaces already to be found in the diagram. Since intersections between two spaces belonging to the same chain are obtained in a trivial way, we only have to look for the intersections given in Table II, in which ** indicates that no general identification can be given.

PROPOSITION 5.2. *Let (A, B) be a standard pair such that B is closed in $\tilde{B}$. Then one has to interpret Table* II *in an obvious way.*

Proof. Referring to the formula describing the intersection of the space in the ith row and jth column as to $[i, j]$, for $1 \leqslant i, j \leqslant 4$ we see that nothing has to be proved if $i \geqslant 3$ and $j \geqslant 3$. Furthermore $[4, 1]$ and $[1, 4]$ are obvious, since $B_A \subseteq B^A$ and $B_G \subseteq B^G$. The remaining results being nontrivial we observe first that the spaces have been listed in increasing order. Since B_0 is in fact the smallest possible space that can arise we see that formulas $[i, j]$, for $i + j \leqslant 4$ will follow from $[2, 3]$ and $[3, 2]$. Before proving these two results let us recall that $[1, 1]$ is part of formula (4.1), and that $B^A{}_G \cap B^G{}_A = \tilde{B}_G \cap \tilde{B}_A = B_0$, thus proving $[2, 2]$, follows from (4.8) combined with (4.4c). Now $[2, 1]$ and $[1, 2]$ follow as mentioned. In order to prove $[3, 2]$ (the arguments for $[2, 3]$ are analog) consider $\mu \in B_A{}^G \cap B^A{}_G$. It will be sufficient to verify that this implies $\mu \in B_A$, because then $[1, 2]$ will imply $\mu \in B_0$. Applying Lemma 4.3 we observe that $\mu \in B_A{}^G$ implies $k * \mu \in B_A$ for all $k \in L^1_w(G)$. But the factorization theorem, applied to $B^A{}_G$ shows that there exists $\mu' \in B^A \subseteq \tilde{B}$ and $k_0 \in L^1_w(G)$ such that $\mu = k_0 * \mu'$. Now let $(e_\alpha)_{\alpha \in I}$ be a bounded approximate unit. Then $(e_\alpha * \mu)_{\alpha \in I}$ is in B_A, but it is also a Cauchy net in $(\tilde{B}, \| \|_\sim)$ (hence $(B_A, \| \|_B)$), since

$$\|e_\alpha * e_{\alpha'} * \mu\|_\sim \leqslant \|e_\alpha * k_0 - e_{\alpha'} * k_0\|_{1,w} \|\mu\|_\sim \to 0 \quad \text{for } \alpha, \alpha' \to \infty.$$

The proof of $[3, 2]$ is thus complete, and all cases where B_0 arises are therefore discussed.

Now only [4.2] and [2, 4] remain. Let us discuss [4, 2]. In view of [4, 1] we only have to show $B^G \cap B^A{}_G \subseteq B_G$. However, using the fact that $k * \mu \in B_G$ for all $k \in L^1_w$ and $\mu \in B^G$ we may show, arguing as above, that $(e_\alpha \mu)_{\alpha \in I}$ is a Cauchy net in B_G, and the proof of the proposition is complete. (We shall refer to $[i, j]$ by writing (5.10i, j)).

Remark 5.2. For the case $B = C_0(G)$ (cf. Example 6.1) it is clear that the four spaces left open in Table II coincide with $C_0{}^A \cap C_0{}^G$. If G is nondiscrete and noncompact this is a space different from the six spaces arising in the diagram. In the same situation one obtains for $L^\infty(G)$ four different space, but the only one not to be found in the diagram (arising in [3, 3]) is again $C_0{}^A \cap C_0{}^G$.

Combining formulas (5.5) and (5.6) with [2, 3] and [3, 2], respectively, we obtain two nontrivial results, showing certain connections between E_5 and E_6.

COROLLARY 5.3. (a) *Assume that any of the equivalent conditions of* (5.5) *is satisfied. Then* E_5 *implies* E_6.

(b) *Assume that any of the equivalent conditions of* (5.6) *are satisfied. Then* E_6 *implies* E_5.

Proof. (a) Assumption (5.5) implies $\tilde{B} = B_G{}^A$. Applying [2, 3] now we obtain from $B^G{}_A = B^G{}_A \cap \tilde{B} = B^G{}_A \cap B_G{}^A$ that $B^G{}_A = B_0$, hence $\{1, 3, 6\} \subseteq E(B)$. In particular, E_6 is satisfied. The reasoning for (b) is quite symmetric.

Remark 5.3. The equivalence of E_5 and E_6 follows from the above results only if $\{1, 2, 3, 4\} \subseteq E(B)$. But this can be seen directly observing that in this case, the upper half of the diagram reduces to one space, i.e., to $\tilde{B}$, and the lower half to B_0. We have not been able to prove equivalence of E_5 and E_6 under (5.5) or (5.6), not even for Segal algebras. It also seems to be difficult to find a counterexample.

Just as a special case of the above result we have the following "practical" version, for which a direct proof would be possible as well:

COROLLARY 5.4. *Assume that* (A, B) *is a standard pair.*

(a) *If* B *is closed in* B^A *and if translation is continuous in* B^A, *then* $B^G \subseteq B^A$ *implies* $B^G{}_A = B_0$. *If furthermore* $B_G = B_0$, *then* $B^G = B = B_G$.

(b) *If* B *is closed in* B^G, *and if* B^G *is an essential A-module containing* B^A, *then* $B^A{}_G = B_0$. *If further* $B_0 = B_A$, *then* $B_A = B = B^A$.

Proof. The assumptions imply that B is closed in $\tilde{B}$, and that $\{1, 3, 5\} \subseteq E(B)$ and $\{2, 4, 6\} \subseteq E(B)$, respectively. By Corollary 5.3 this implies $\{1, 3, 5, 6\}$ and $\{2, 4, 5, 6\} \subseteq E(B)$, respectively. The assertions as well as the additional equalities are then seen to be true by looking at the diagram.

Under certain circumstances it is very easy to check that $B^A = \tilde{B}$ or $B^G = \tilde{B}$, without calculating both of them.

PROPOSITION 5.5. *Let (A, B) be a standard pair such that B is closed in $\tilde{B}$.*

(A) *Assume that $B \subseteq A = A^G$. Then B satisfies E_1 and E_3.*

(B) *Assume that $B \subseteq L^1_w$ and that B is a right L^1_w-module or that G is a* [SIN]*-group. Then B satisfies E_2 and E_4.*

Proof. We show (B), the proof of (A) being easier. By (5.6) we have to show $B^A{}_G \subseteq B_0$. Let $\mu \in B^A{}_G$ be given, and let $(e_\alpha)_{\alpha \in I}$ be a bounded approximate unit for $L^1_w(G)$ in A_0. Then we have $\lim_\alpha \| e_\alpha * \mu - \mu \|_{B^A} = 0$, hence convergence in $\tilde{B}$. Since

$$e_\alpha * \mu \in A_0 * L^1_w \subseteq B_0 * L^1_w \subseteq B_0$$

(this is clear, cf. Lemma 3.1) the limit, i.e., μ, belongs to B_0. If G is a [SIN]-group we may use instead a central approximate unit $(e_\alpha)_{\alpha \in I}$ in $L^1_w \cap B_0$, which is an (abstract) Segal algebra in $L^1_w(G)$ (cf. [18]).

Remark 5.4. The above result implies among other $B^G = \tilde{B}$ for symmetric Segal algebras or Segal algebras on [SIN]-groups (cf. [9, 10]).

As we have already observed (cf. Remark 3.6) any two spaces B^1 and B^2 in standard situation form a compatible pair. Therefore it makes sense to consider $B^1 \cap B^2$ and $B^1 + B^2$. These are known to be Banach spaces with respect to their natural norms, i.e., for

$$\|f\|_\wedge := \|f\|_{B^1} + \|f\|_{B^2} \text{ and}$$
$$\|f\|_\vee := \inf\{\|f^1\|_{B^1} + \|f^2\|_{B^2}, f = f^1 + f^2, f^1 \in B^1, f^2 \in B^2\}$$

(cf. [2, Sect. 2.3]). Now looking for the compatibility of these constructions with those discussed in Sections 3, 4 we obtain

THEOREM 5.6. *Given two standard pairs (A, B^1) and (A, B^2) one has*

(A) *$(A, B^1 \cap B^2)$ and $(A, B^1 + B^2)$, each of the spaces endowed with its natural norms, are standard pairs as well.*

(B) *The (right) standard pair $(A, B_0^{1'} + B_0^{2'})$ is the dual pair of $(A, B^1_0 \cap B^2_0)$.*

(C) *The following formulas hold true*:

$$(B^1+B^2)_0 = B^1_0 + B^2_0, \tag{5.12a}$$

$$(B^1+B^2)^{\sim} = (B^1)^{\sim} + (B^2)^{\sim}, \tag{5.12b}$$

$$(B^1 \cap B^2)_0 = B^1_0 \cap B^2_0, \tag{5.13a}$$

$$(B^1 \cap B^2)^{\sim} = (B^1)^{\sim} \cap (B^2)^{\sim}, \tag{5.13b}$$

$$(B^1 \cap B^2)_A = B^1_A \cap B^2_A, \tag{5.14a}$$

$$(B^1 \cap B^2)^A = (B^1)^A \cap (B^2)^A, \tag{5.14b}$$

$$(B^1 \cap B^2)_G = B^1_G \cap B^2_G, \tag{5.15a}$$

$$(B^1 \cap B^2)^G = (B^1)^G \cap (B^2)^G. \tag{5.15b}$$

Proof. (A) is obvious, and (B) is essentially a consequence of [2, Sect. 2.7]. We only discuss (5.12) and (5.13), the other results being clear or following by similar arguments, (cf. Theorem 1.1(A)). The nontrivial inclusions are shown as follows:

Using Lemma 3.4 (Eq. (3.6a)) in both directions we have for $f \in B^1_0 \cap B^2_0$ $\|R_\eta f - f\|_{B^1} \to 0$ and $\|R_\eta f - f\|_{B^2} \to 0$ for $\eta \to \infty$. This implies of course convergence in $B^1 \cap B^2$, hence $f \in (B^1 \cap B^2)_0$. Given $f \in (B^1+B^2)_0$ we recall that $f = k * hf'$ for some $f' \in B^1+B^2$, $k \in L^1_w(G)$ and $h \in A(G)$ by Remark 4.2. Since $f' = f^{1'} + f^{2'}$, with $f^i \in B^i$, we have $f = f^1 + f^2 \in B^1_0 + B^2_0$, since $f^i := k * hf^{i'} \in B^i_0$ for $i = 1, 2$. While (5.13b) is obvious, if one uses Lemma 3.4 (Eq. (3.6b)), we have to discuss the inclusion $(B^1+B^2)^{\sim} \subseteq (B^1)^{\sim} + (B^2)^{\sim}$. Let $\mu \in (B^1+B^2)^{\sim}$ be given. Then we have $\mu = \sigma\text{-}\lim_\eta R_\eta \mu \cdot (R_\eta \mu)_{\eta \in M}$ being bounded in B^1+B^2. Therefore one can find bounded nets $(f^i_\eta)_{\eta \in M}$ in B^i, $i = 1, 2$, such that $R_\eta \mu = f^1_\eta + f^2_\eta$ for all $\eta \in M$. The nets being equicontinuous in (A_0', σ) one can find subnets which are σ-convergent to $\mu_i \in (B^i)^{\sim}$ for $i = 1, 2$. Then $\mu_1 + \mu_2 \in (B^1)^{\sim} + (B^2)^{\sim}$, but $\mu_1 + \mu_2 = \sigma\text{-}\lim_\eta f^1_\eta + \sigma\text{-}\lim_\eta f^2_\eta = \sigma - \lim_\eta f^1_\eta + f^2_\eta = \sigma\text{-}\lim_\eta R_\eta \mu = \mu$. All relevant assertions of the theorem are now proved.

Remark 5.5. We conclude this section by mentioning that interpolation (real or complex) between two Banach spaces in the standard situations gives again standard spaces. If one of the two spaces is an essential module, then the interpolation space is essential as well for all $\Theta \in (0, 1)$.

Remark 5.6. It is interesting that $(B^1+B^2)^A$ is not equal to $(B^1)^A + (B^2)^A$ in general, nevertheless (5.12b) holds true.

Remark 5.7. If B is a space in standard situation one can define in a natural way the support of elements in B (cf. [12]). Define

$B_K = \{\mu \in B \mid \operatorname{supp} \mu \subseteq K\}$, K a compact subset in G, then B_K is a closed subspace of B. Now if $B^A = \tilde{B}$, then using Theorem 3.8 it can easy be seen that B_K is w^*-closed in $\tilde{B}$. Using standard arguments from functional analysis it follows that B_K is isomorph to a dual Banach space.

6. Applications and Concluding Remarks

In this section we can only give some ideas how to work with the above results, how to establish the diagram corresponding to a given space, and how to draw new informations from known results, making use of our formulas. Various further examples of spaces in standard situation are given in [12, Sect. 1].

Example 6.1. We start with the discussion of $B = C^0(G)$, G a noncompact and nondiscrete group. Here we may take $A = C^0(G)$ and $w \equiv 1$. It is clear that $(C^0)^A = C^b(G)$, and since $\mathscr{K}(G)$ is dense in $C_0(G)$ equalities E_3 and E_4 are satisfied. Hence at most 6 spaces can arise in the diagram. Since L^∞ is apparently a dual module (of $L^1(G)$) we have $\tilde{B} \subseteq (L^\infty) \cong L^\infty$ satisfying $L^1 * L^\infty \subseteq C^b(G) = B^A$. Hence $L^\infty \subseteq (B^A)^G = \tilde{B}$. The spaces $B^G{}_A = \tilde{B}_A$ and $B^A{}_G = \tilde{B}_G$ are now easily seen to be just $L_0^\infty = \{f \mid f \in L^\infty(G), f$ vanishes at infinity$\}$, and $C^{\text{lub}}(G)$ (left uniformly continuous, bounded functions). The space $(C^0)^G = (L^1, C^0)$ can be identified (cf. [9, Theorem 3.5] or (4.3b)) with $\{f \mid f \in L^\infty(G), \chi_K * f \in C_0(G)$ for each compact set $K \subseteq G\}$. By considering $f = \sum L_{y_n} f_n$, where $\|f_n\|_\infty = 1$, $\operatorname{supp} f_n \subseteq K_0, f_n \to 0$ in the vague topology, and $y_n \to \infty$ sufficiently fast one can see that L_0^∞ is a proper subspace of $(C^0)^G$, i.e., $6 \notin E(B)$.

Since it is clear that the other spaces do not coincide we have $E(C^0(G)) = \{3, 4\}$. Thus $C^0(G)$ represents the "general case" of a space containing the test functions as a dense subspace (hence satisfying E_3 and E_4, cf. Theorem 4.2).

Example 6.2. Starting now with B being any of the five other spaces appearing in the diagram the reader will find that again all six spaces arise. In fact, all these spaces have of course the same completion ($L^\infty(G)$), and the same kernel ($C^0(G)$). The remaining 4 spaces being derived from these two spaces by "elementary" operations (e.g., $C^b(G) = (C_0)^A$) there have to be at least these 6 spaces. On the other hand there cannot arise a new space due to the reduction method (Theorem 4.7(A)). Checking directly, or better only working with the symbols, assuming that $E(B) = \{3, 4\}$ one can prove that $E(\tilde{B}) = \{1, 2\}$, $E(B^G{}_A) = \{1, 4\} = E(B_A{}^G)$, and $E(B^A{}_G) = \{2, 3\} = E(B_G{}^A)$.

Example 6.3. The fact that $E(C^0(G)) \cap E(L^\infty(G)) = \varnothing$ shows already that no general statement about further equalities can be made for the general situation. By "combining" these to examples to $B := \{f \mid f \in L^\infty(\mathbb{R}),$ f continuous on $[0, \infty)$, $\lim_{x\to\infty} |f(x)| = 0\}$ one can even obtain a space for which the "full" diagram appears, i.e., for which $E(B) = \varnothing$, and ten different spaces arise. Combining some of the other spaces of the diagram in a similar way one obtains spaces satisfying exactly one of the equalities E_1–E_4.

EXAMPLE 6.4. Let us now look for the diagram associated with a Beurling algebra $L^1_w(G)$ (take $A = C^C(G)$). Since it is a two-sided convolution module over itself we know (applying Proposition 5.5 and Theorem 3.8) that $(L^1_w)^G = (L^1_w)^{\sim} = (L^1_w)'_0{}' = (L^\infty_{w^{-1}})_0{}' = (C^0_{w^{-1}})' = M^1_w(G) :=$ $\{\mu \mid \mu w \in M(G)\}$. This is a result due to Gaudry (cf. [14]). Since $\mathscr{K}(G)$ is dense in L^1_w, and $M^1_w(G)$ is apparently an essential $C^0(G)$-module, we have therefore $\{2, 3, 4, 6\} \subseteq E(L^1_w)$, hence E_5 is also satisfied by Corollary 5.3(b) (or 5.4(b)). Thus $E(L^1_w) = \{2, 3, 4, 5, 6\}$ whenever G is nondiscrete.

EXAMPLE 6.5. Let us now look for the spaces $L^1_w \cap L^p(G)$, $1 < p < \infty$. $L^p(G)$ being a reflexive space we have $L^p = (L^p)^{\sim}$ for $1 < p < \infty$. Using (5.13b) we obtain $(L^1_w \cap L^p)^{\sim} = (L^1_w)^{\sim} \cap L^p = M^1_w \cap L^p = L^1_w \cap L^p$. Therefore the diagram reduces to one spaces in that case, although these spaces cannot be reflexive Banach spaces (for noncompact G). According to Theorems 3.8 and 5.6 their predual is just $C^0_{w^{-1}} + L^{p'}$, $1/p + 1/p' = 1$. Since $(L^1_w \cap L^p)^A = L^1 \cap L^p$, we have $E(L^1_w \cap L^p) = \{2, 3, 4, 5, 6\}$.

EXAMPLE 6.6. The spaces in Example 6.5 are typical examples of (abstract) Segal algebras in a Beurling algebra (for $w \equiv 1$ one has the usual Segal algebras as introduced by Reiter). Applying the arguments of the proof for Proposition 5.5 one verifies that a Banach space $(B, \| \|_B)$ in standard situation is a Segal algebra in $L^1_w(G)$ if and only if $B \subseteq L^1_w(G)$ and A_0 is dense in B. Therefore, a (usual) Segal algebra in standard situation is pseudosymmetric (symmetric) if and only if S is (isometrically) right invariant, i.e., $R_y S \subseteq S$ for all $y \in G$ (and $\|R_y f\|_S = \|f\|_S$ for all $f \in S$). In fact, continuity of right translation in A_0 then implies the same for $(S, \| \|_S)$, and since A_0 contains positive functions with small support the same is true for S. In particular, symmetry implies pseudosymmetry for Segal algebras in standard situation (since "most" spaces are in standard situation the nomenclature therefore appears to be justified, although there are examples of symmetric Segal algebras which are not pseudosymmetric; cf. [24] for the terminology).

EXAMPLE 6.7. As mentioned above the diagram for Segal algebras cannot consist of more than 4 spaces. To see that this can actually be the case consider $L^1 \cap C^0(G) =: B$ on a noncompact, nondiscrete group. Then $\tilde{B} = (L^1)^{\sim} \cap (C^0)^{\sim} = M(G) \cap L^\infty(G) = L^1 \cap L^\infty(G)$ by (5.13b). The spaces $B^G{}_A = L^1 \cap L_0^\infty$ and $B^A = (L^1)^A \cap (C_0)^A = L^1 \cap C^b(G)$ are of course different from B and $\tilde{B}$ in the case. Therefore $E(L^1 \cap C^0) = \{2, 3, 4\}$.

EXAMPLE 6.8. In order to show Segal algebras for which exactly two different spaces arise in the diagram let us consider the following spaces on locally compact Abelian groups. Not using the usual symbol, we define $F_p(G) := \{f \mid f \in L^1, \hat{f} \in L^p(\hat{G})\}$, $1 \leqslant p < \infty$. It is not difficult to check that F_p is a Banach space (with respect to the norm $\|f\|_{F_p} := \|f\|_1 + \|\hat{f}\|_p$). Using the extended Fourier transform $\mathscr{F}$ (cf., [11]) we write $F_p(G) = L^1(G) \cap \mathscr{F}L^p(G)$. $\mathscr{F}L^p(G)$ being a strongly character invariant, homogeneous Banach space of quasimeasures (cf. [12], note that $S_0(G)$ is dense in this space, because $L^p(\hat{G})$ contains $S_0(\hat{G})$ as a dense subspace) it is in standard situation with respect to the Fourier algebra $A(G)$, satisfying E_3 and E_4. Therefore $F_p(G)$ satisfies E_3 and E_4 as well ((5.14a), (5.15a)), i.e., $F_p(G)$ is a Segal algebra. Using (5.13b) and Proposition 5.5 we obtain $F_p^G = F_p^{\sim} = M^1(G) \cap \mathscr{F}L^p(G) = \{\mu \mid \mu \in M(G),\ \hat{\mu} \in L^p(G)\}$. $F_p^{\sim}$ being apparently an essential $A(G)$-module we have E_5, and again by Corollary 5.3(b) E_6, i.e., $\{2, 3, 4, 5, 6\} \subseteq E(F_p)$. Therefore only F_p and $F_p^{\sim}$ remain. One can show that the inclusion is proper (if and) only if $p > 2$ (and if G is appropriate). Referring to Corollary 3.10 we can state $F_p = F_p^{\sim}$ if and only if F_p is a dual of (essential) standard space. In that case one has $E(F_p) = \{1, 2, 3, 4, 5, 6\}$. For $F_p \neq F_p^{\sim}$ one has exactly $E(F_p) = \{2, 3, 4, 5, 6\}$.

The same arguments, without relevant changes apply if $L^1(G)$ is replaced by $L^1_w(G)$, and if $L^p(\hat{G})$ is replaced by a Lorentz space $L(p, q)$, an Orlicz space $L^\Phi(\hat{G})$, or a weighted space $L^p_w(\hat{G})$, if the weight satisfies the condition of Beurling–Domar (cf. [23], $A := A_w(G) := \{\hat{f} \mid f \in L^1_w(\hat{G})\}$ is then a "nice" algebra). Such spaces have found much attention in the literature (cf. [10, 19, 20] for further references).

EXAMPLE 6.9. As mentioned above A_0 is always dense in a Segal algebra in standard situation, and $\tilde{B} = B^G$ by Proposition 5.5. Therefore $\{2, 3, 4\} \subseteq E(S)$ for any Segal algebra. In view of Corollary 5.3 this gives 5 possible forms of $E(S)$, depending which of the equalities E_1, E_5 or E_6 are satisfied. Three different situations are given in the above examples, but we have not found Segal algebras for which $E(S) = \{2, 3, 4, 5\}$ or $E(S) = \{1, 2, 3, 4\}$. It would be of interest to know whether one of these formal possibilities can be excluded by a proof.

EXAMPLE 6.10. We conclude the list of applications by a result concerning the A_p-algebras, $1 < p < \infty$, due to Herz see [6]. We suppose that G is amenable. In that case $A_p(G)$ is known to be a "nice" Banach algebra with respect to pointwise multiplication, satisfying (A1)–(A5). We consider the standard pair (A_p, A_p). Therefore it is clear that A_p satisfies E_3 and E_4. Denoting $A_p^{A_p}$ by W_p it follows from Proposition 5.5(A) that $W_p = (A_p)^{\sim}$, i.e., E_1 holds. According to a result of Cowling ([6]) translation is continuous in W_p, i.e., ${A_p}^{A_p}{}_G = {A_p}^{A_p}$, or E_5 is satisfied. Now, applying Corollary 5.3 we see that $A_p^G = A_p$, i.e., A_p is complete as an $L^1(G)$-module. In particular, $\sup \|e_\alpha * \mu\|_{A_p} < \infty$ implies $\mu \in A_p(G)$ for any bounded approximate unit in $L^1_w(G)$.

It would be possible to continue with a long list of further examples, showing also connections between certain results in the literature. Instead, we conclude with several remarks.

A list of spaces in standard situation and corresponding algebras A is given in [12]. We only indicate that it contains many spaces arising in harmonic analysis, Euclidean Fourier analysis and distribution theory, including various types of generalized Lipschitz spaces as well as their duals, and also spaces of ultradistributions (on lca. groups, cf. [3, 27]).

For the class of strongly character invariant Banach spaces of quasimeasures on an lca. group G (mentioned in [12, Sect. 1]) the symmetry between $A(G)$ and $L^1(G)$ action (a left–right symmetry of the diagram) can be made even more explicit, by observing that the Fourier transform (in the sense of $S_0'(G)$) maps the diagram of B bijective on the diagram of $\mathscr{F}B$, interchanging left and right (i.e., convolution and multiplication).

Theorem 5.8, showing that $\tilde{B}$ is a dual space can be used to show that spaces of multipliers (i.e., the corresponding spaces of convolution kernels) can often be represented as duals of Banach spaces in standard situation. Thus one can show that $H_G(S, S)$, the space of multipliers on a Segal algebra in standard situation, coincides with the dual of a Banach space of continuous functions. This extends a result due to Krogstad concerning Segal algebras on Abelian groups satisfying property P. (cf. [17]).

Some of the results could have been proved under slightly more general conditions, but at the cost of more technical problems. In particular, it is not always necessary to assume the existence of bounded approximate units in A. It is intended to come back to this problem in a subsequent paper.

Several results (in particular concerning Section 3) can be formulated by means of the theory of generalized L^1-algebras as considered by Leptin (see [21]). Details concerning this different approach are to be given elsewhere.

Under suitable assumptions one can also show that an isomorphism between standard pairs (respecting both structures) can be shown to be always induced from an isomorphism of the underlying groups.

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REFERENCES

1. G. F. BACHELIS, W. A. PARKER AND K. A. ROSS, Local units in $L^1(G)$, *Proc. Amer. Math. Soc.* **31** (1972), 312–313.
2. J. BERGH AND J. LÖFSTRÖM, "Interpolation Spaces," Grundl. Math. Wiss.," Vol. 223, Springer-Verlag, Berlin, 1976.
3. G. BJÖRCK, Linear partial differential operators and generalized distributions, *Ark. Mat.* **6** (1966), 351–407.
4. N. BOURBAKI, "Éléments de mathématique, Intégration," Chaps I–IV, 2nd ed., Hermann, Paris, 1965.
5. W. BRAUN, "Segalalgebren," Diplomarbeit, Univ. Heidelberg, 1981.
6. M. COWLING, An application of Littlewood–Paley theory in harmonic analysis, *Math. Ann.* **241** (1979), 84–96.
7. R. S. DORAN AND J. WICHMANN, "Approximate identities and factorization in Banach modules", Lecture Notes Math. No. 768, Springer-Verlag, Berlin, 1979.
8. R. E. EDWARDS, "Functional analysis: Theory and Applications," Holt, Rinehart & Winston, New York, 1975.
9. H. G. FEICHTINGER, Multipliers from $L^1(G)$ to a homogeneous Banach space, *J. Math. Anal. Appl.* **61** (1977), 341–356.
10. H. G. FEICHTINGER, Results on Banach ideals and spaces of multipliers, *Math. Scand.* **41** (1977), 315–324.
11. H. G. FEICHTINGER, Un espace de distributions temperées sur les groupes localement compacts abélians, C. R. *Acad. Sci. Paris Ser. A* **290** (17) (1980), 791–794.
12. H. G. FEICHTINGER, "Compactness in Translation Invariant Banach Spaces of Distributions and Compact Multipliers, *J. Math. Anal. Appl.*, to appear.
13. H. G. FEICHTINGER AND M. LEINERT, Individual Factorization in Banach Modules, preprint.
14. G. I. GAUDRY, Multipliers of weighted Lebesgue and measure spaces, *Proc. London Math. Soc.* (2) **19** (1969), 327–340.
15. M. GROSSER, "Bidualräume und Vervollständigungen von Banachmoduln," Lecture Notes Math. No. 717, Springer-Verlag, Berlin, 1979.
16. S. KAIJSER, A note on Dual Banach spaces, *Math. Scand.* **41** (1977), 525–530.
17. H. E. KROGSTAD, Multipliers of Segal algebras, *Math. Scand.* **38** (1976), 285–303.
18. E. KOTZMANN AND H. RINDLER, Central approximate units in ideals of $L^1(G)$, *Proc. Amer. Math. Soc.* **57** (1976), 155–158.
19. R. LARSEN, "An Introduction to the Theory of Multipliers," Grundl. Math. Wiss. Vol. 175, Springer-Verlag, Berlin, 1971.
20. R. LARSEN, The algebras of functions with Fourier transform in L_p, a survey; Nieuw Arch. Wisk. (3) **22** (1974), 195–246.
21. H. LEPTIN, Verallgemeinerte L^1-Algebren und projektive Darstellungen lokal kompakter Gruppen, *Invent. Math.* **3** (1967), 257–281.
22. J. LIUKKONEN AND R. MOSAK, Harmonic analysis and centers of group algebras, *Trans. Amer. Math. Soc.* **195** (1974), 147–163.

23. H. Reiter, "Classical Harmonic Analysis and Locally Compact Groups," Oxford Univ. Press, London/Oxford, 1968.
24. H. Reiter, "L^1-Algebras and Segal Algebras," Lecture Notes Math. No. 231, Springer, Berlin, 1971.
25. M. A. Rieffel, Induced representations of Banach algebras and locally compact groups, *J. Funct. Anal.* **1** (1967), 443–491.
26. H. H. Schaefer, "Topological Vector Spaces," Graduate Texts in Mathematics, Vol. 3, Springer-Verlag, Berlin, 1971.
27. H. Triebel, "Fourier Analysis and Function Spaces," Teubner, Leipzig, 1977.